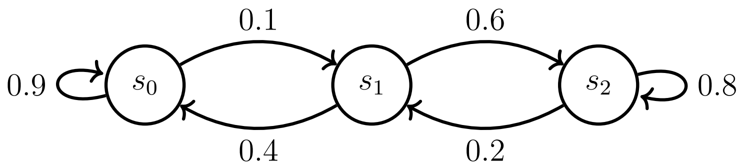
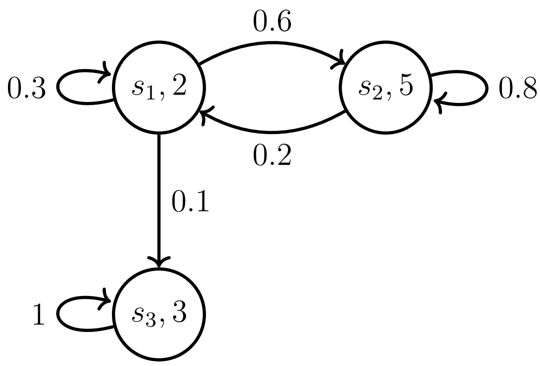
Quantitative Verification – Exercise sheet 7

Exercise 7.1
Compute the steady state distribution of the following Markov Chain by solving the corresponding linear equation system.

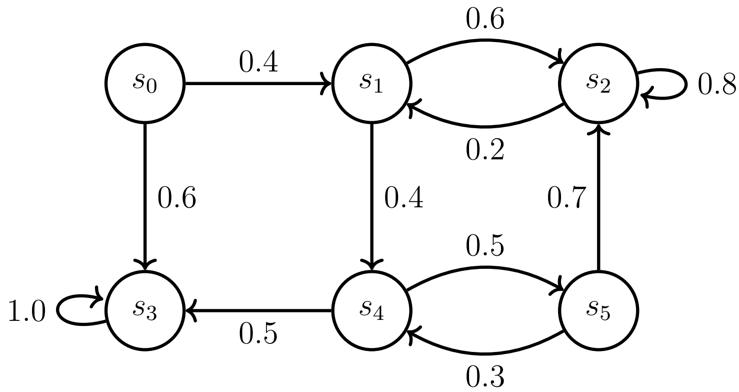


Exercise 7.2
For each initial distribution π, compute the average reward obtained in the following Markov Chain.

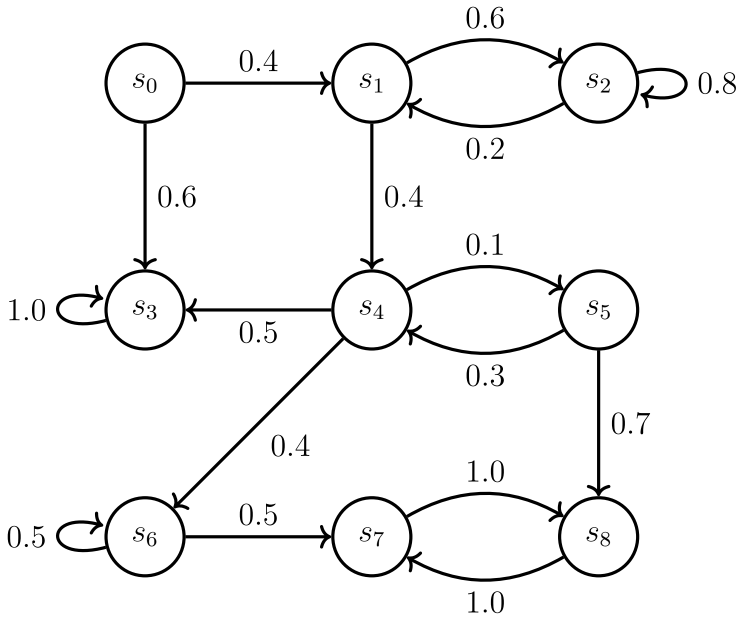


Given an initial distribution π and step bound n, how can you compute the average (accumulated) n-step reward? What is the 2-step average reward for $\pi = \{s_1 \to 1\}$?

Exercise 7.3
Compute the probability of reaching the set $\{s_1, s_4\}$ in the following Markov Chain.



Exercise 7.4
For the following Markov Chain, determine the set of its strongly connected components and identify which are "bottom".



Solution 7.1
First, we need the transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Now, we solve the equation $\pi = \pi P$ to obtain the steady state distribution. This equation is equivalent to $\pi P - \pi = \bar{0}$, i.e. $\pi(P - I_3) = \bar{0}$, where I_3 is the 3-dimensional identity matrix. We have that

$$P - I_3 = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.4 & -1 & 0.6 \\ 0 & 0.2 & -0.2 \end{bmatrix}$$

and can solve the resulting equation system using, e.g., Gauss elimination:

$$-0.1x_1 + 0.4x_2 = 0$$
$$0.1x_1 - x_2 + 0.2x_3 = 0$$
$$0.6x_2 - 0.2x_3 = 0$$

This gives, for example:

$$x_1 = 4x_2$$
$$x_3 = 3x_2$$

Since π has to be a distribution, we get the unique solution $\pi = [\frac{1}{2}, \frac{1}{8}, \frac{3}{8}]$.

Solution 7.2
Eventually, any run ends up in s_3, hence the average (expected) reward will be 3 in all cases.

To compute the average (expected) n-step reward, we can compute the total expected reward and divide by n. To obtain the total expected reward, we compute the distribution for each step:

$$\pi_0 = \{s_1 \to 1\}$$
$$\pi_1 = \{s_1 \to 0.3, s_2 \to 0.6, s_3 \to 0.1\}$$
$$\pi_2 = \{s_1 \to 0.21, s2 \to 0.66, s_3 \to 0.13\}$$

Hence, the rewards obtained in each step are:

$$r_0 = 1 \cdot 2$$
$$r_1 = 0.3 \cdot 2 + 0.6 \cdot 5 + 0.1 \cdot 3 = 3.9$$
$$r_2 = 0.21 \cdot 2 + 0.66 \cdot 5 + 0.13 \cdot 3 = 4.11$$

In total, we have a 2-step expected reward of $2+3.9+4.11 = 10.01$, and the average reward equals $3.33\bar{6}$.

Solution 7.3
We have the set with zero reachability probability, $S_{=0} = \{s_3\}$ so we remove $S_{=0}$ from our reachability equation $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$.

$$\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0.7 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.4 \\ 0.2 \\ 0.3 \end{bmatrix}$$

Reordering gives,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & -0.7 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.3 \end{bmatrix}$$

which gives us $x_0 = 0.4, x_2 = 1$ and $x_5 = 1$.

Solution 7.4
Connected components:

- $\{s_0\}$
- $\{s_1, s_2\}$
- $\{s_3\}$
- $\{s_4, s_5\}$
- $\{s_6\}$
- $\{s_7, s_8\}$

Bottom connected components are $\{s_3\}$ and $\{s_7, s_8\}$.