# Quantitative Verification

## Chapter 5: Continuous-time Markov chains

Jan Křetínský

Technical University of Munich

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# Continuous-time Markov chains CTMC

# CTMC - Motivation

Dicrete-time abstraction is fitting for situations where

- flow of time irrelevant: execution steps, steps of a game, ...
- flow of time important but timing within the steps irrelevant: discretized time (e.g. one day per step) without loss of precision.

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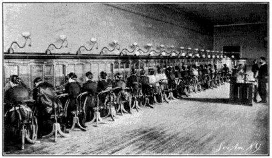

How long do I need to wait (in 1910') on average for a telephone connection?

How many pump machines a gas station needs to satisfy the peak demand?

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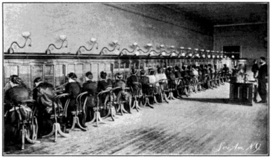

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How often do safety functions in a nuclear power plant fail (at the same time)?

- One could address these questions by discretizing time.
- However, continuous-time models are more suitable.
- Continuous-time models are actually also easy to solve!

# CTMC – Stochastic Process Definition

# CTMC - Math / Statistics Definition (1)

## Recall

A discrete-time stochastic process $\{X_n \mid n \in \mathbb{N}\}$ over state space $S$ is:

- Markov if for all $n > 1$ and $s_0, \ldots, s_n$ with $P(X_{n-1} = s_{n-1}) > 0$:

  $$P(X_n = s_n \mid X_{n-1} = s_{n-1}, \ldots, X_0 = s_0) = P(X_n = s_n \mid X_{n-1} = s_{n-1}).$$

- homogeneous if for all $n > 1$ and $s, s' \in S$ with $P(X_0 = s) > 0$:

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  $$P(X_{n+1} = s' \mid X_n = s) = P(X_1 = s' \mid X_0 = s)$$

## Definition:

A continuous-time stochastic process $\{X_t \mid t \in \mathbb{R}_{\geq 0}\}$ over states $S$ is

- Markov if for all $n > 1$, $0 = t_0 < t_1 < \cdots < t_n$ and $s_0, \ldots, s_n$ with $P(X_{t_{n-1}} = s_{n-1}) > 0$:

  $$P(X_{t_n} = s_n \mid X_{t_{n-1}} = s_{n-1}, \ldots, X_{t_0} = s_0) = P(X_{t_n} = s_n \mid X_{t_{n-1}} = s_{n-1}).$$

- homogeneous if for all $t, t' \in \mathbb{R}$ and $s, s' \in S$ with $P(X_0 = s) > 0$:

  $$P(X_{t+t'} = s' \mid X_t = s) = P(X_{t'} = s' \mid X_0 = s).$$

We consider only discrete-space homogeneous Markov processes, that we call continuous-time Markov chains (CTMC).

# CTMC - Math / Statistics definition (2)

## Sojourn time:

Let $\{X_t \mid t \in \mathbb{R}_{\geq 0}\}$ be a continuous-time Markov chain. We define for each state $s$ and $i \in \mathbb{N}$

- random variables $A_{s,i}$, $B_{s,i}$ denoting time of entering and leaving $s$ for the $i$-th time, respectively i.e. $A_{s,1} = \inf\{t \geq 0 \mid X_t = s\}$, and $B_{s,i} = \inf\{t > A_{s,i} \mid X_t \neq s\}$ and $A_{s,i+1} = \inf\{t > B_{s,i} \mid X_t = s\}$;
- random variable $T_{s,i}$ denoting the sojourn time upon the $i$-th visit to $s$, i.e. $T_{s,i} = B_{s,i} - A_{s,i}$.

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## Observation:

For any $i$ and $t, t'$ we have from the two properties

$$P(T_{s,i} \leq t + t' \mid T_{s,i} > t) = P(T_{s,i} \leq t').$$

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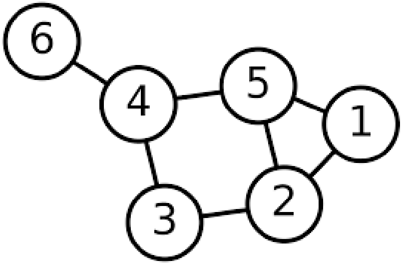
## Proposition:

Each $T_{s,i}$ is exponentially distributed with some rate $\lambda_s$, i.e.

$$F_{T_{s,i}}(x) = 1 - e^{-\lambda_s x} \qquad f_{T_{s,i}}(x) = \lambda_s e^{-\lambda_s x}$$

with the expected value given by $E[T_{s,i}] = \frac{1}{\lambda_s}$.

# CTMC – Graph Based Definition



# CTMC [Graph] - Definition (1)

### Definition: CTMC

A continuous-time Markov chain is a tuple $(S, \mathrm{R}, \pi_0)$ where

- $S$ is the set of states,
- $\mathrm{R} : S \times S \to \mathbb{R}_{\geq 0}$ is the transition rate matrix, and
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## Definition: Embedded DTMC

We define the exit rate of a state $s \in S$ as

$$E(s) = \sum_{s' \in S} \mathrm{R}(s, s') \quad \leftarrow \text{including the self-loop!}$$

The embedded DTMC coincides on $S$ and $\pi_0$ but has the transition probability matrix $\mathrm{E}$ defined by

$$\mathrm{E}(s, s') = \begin{cases} \frac{\mathrm{R}(s,s')}{E(s)} \text{ if } E(s) > 0, \\ 0 \text{ if } E(s) = 0 \wedge s \neq s', \textit{and} \\ 1 \text{ if } E(s) = 0 \wedge s = s'. \end{cases}$$

# CTMC [Graph] - Definition (2)

Definition:

The probability measure for a CTMC $(S, \mathrm{R}, \pi_0)$ is induced by the measure for cylinder sets $P(C(s_0 I_0 \ldots s_n))$ defined as

$$\pi_0(s) \prod_{0 \leq i < n} \mathrm{E}(s_i, s_{i+1}) \left( e^{-E(s_i) \inf I_i} - e^{-E(s_i) \sup I_i} \right).$$

# CTMC - Solution Techniques

# CTMC - Transient Analysis (1)

## Symbolic Solution:

For a CTMC $\mathcal{C} = (S, \mathrm{Q}, \pi_0)$, the transient probability distribution $\pi_t$ at time $t$ satisfies

$$\frac{d\pi_t}{dt} = \pi_t \mathrm{Q}.$$

We can symbolically solve this system of differential equations by

$$\pi_t = \pi_0 e^{\mathrm{Q}t}$$

where

$$e^{\mathrm{Q}t} = \sum_{i=0}^{\infty} \frac{(\mathrm{Q}t)^i}{i!}.$$

But unfortunately $(\mathrm{Q}t)^i$ is unstable to compute and the infinite sum is not easy to truncate.

# CTMC - Transient Analysis (2)

Another approach?

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Another approach?

Separate the discrete and continuous randomness!

# CTMC - Uniformization (1)

### Definition: Uniformization Rate

For a CTMC $\mathcal{C} = (S, \mathrm{R}, \pi_0)$ the uniformization rate $q$ is defined as

$$q \geq \max_{s \in S} E(s).$$

### Definition: Uniformized DTMC

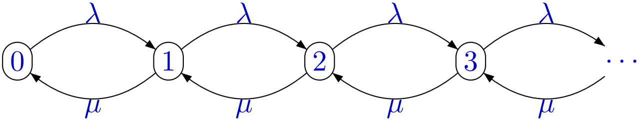
For a CTMC $\mathcal{C} = (S, \mathrm{R}, \pi_0)$, the uniformized DTMC (with uniformization rate $q$), denoted by $uni(\mathcal{C})$ is defined as the DTMC $(S, \mathrm{P}, \pi_0)$ with

$$\mathrm{R}'(i,j) = \begin{cases} \mathrm{R}(i,j) & \text{for } i \neq j \\ q - \sum_k \mathrm{R}(i,k) & \text{for } i = j \end{cases}.$$

# CTMC - Uniformization (2)

Example:

CTMC:

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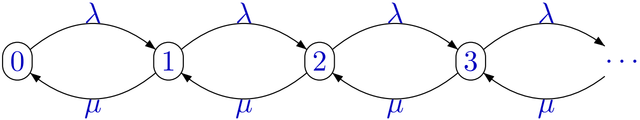


Uniformized DTMC with $q = \lambda + \mu$: add a self-loop with rate $\mu$ on 0

# CTMC - Transient Analysis - By Uniformization

**Lemma:**
Let $\mathcal{C} = (S, \mathrm{R}, \pi_0)$ be a CTMC and $uni(\mathcal{C}) = (S, \mathrm{P}, \pi_0)$ its uniformized DTMC with uniformization rate $q$ and $t > 0$.
Let $\psi(i, qt)$ denote the Poisson probability at $i$ (with parameter $qt$) and $\pi'_i$ the transient probability distribution of the uniformized DTMC at time step $i$. Then

$$\pi_t = \sum_{i=0}^{\infty} \psi(i, qt)\pi'_i.$$

# CTMC - Transient Analysis - Algorithm

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How to compute the infinite sum $\sum_{i=0}^{\infty} \pi'_i \psi(i, qt)$:

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- For each $\epsilon > 0$, the Fox-Glynn algorithm provides bounds $L, R$ such that

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  Moreover, $\psi(i, qt)$ for $L \leq i \leq R$ can be computed efficiently.
- Computation of $\pi'_i$ via the known algorithms for DTMCs.

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By taking higher uniformization rate than $\max_s E(s)$, we have self-loops in every state; hence the uniformized DTMC is aperiodic (other conditions on existence of steady-state were equivalent).

# Continuous Stochastic Logic (CSL)

# CSL - Labelled CTMC and Syntax

## Definition: Labelled CTMC

A labelled CTMC is a tuple $\mathcal{C} = (S, \mathrm{R}, \pi_0, L)$ with labelling function $L : S \rightarrow 2^{AP}$ where $AP$ is a set of atomic propositions.

## Definition: Syntax of CSL

State formulas:

$$\Phi = true \mid a \mid \Phi_1 \wedge \Phi_2 \mid \neg\Phi \mid \mathcal{P}_J(\phi) \mid \mathcal{S}_J(\Phi)$$

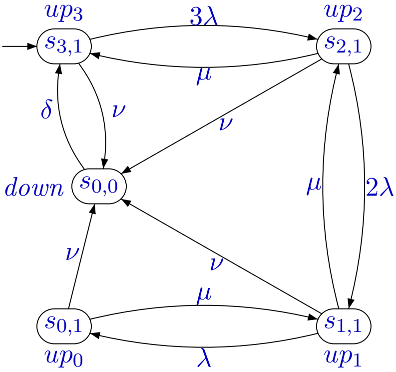
where $a \in AP$, $J \subseteq [0, 1]$ is an interval with rational bounds.
Path formulas:

$$\phi = \mathcal{X}^{\,I}\Phi \mid \Phi_1\, \mathcal{U}^{\,I}\Phi_2$$

where $I \subseteq \mathbb{R}_{\geq 0}$ denotes an interval.

## Example:

We consider a triple modular redundant system:



# CSL - Example

## Specifications:

Let $up = \neg down$, then we can specify the following performance properties:

- $\mathcal{S}_J(up)$: steady state availability
- $\mathcal{P}_J(\mathcal{F}^{[t,t]} up)$: instantaneous availability at time $t$
- $\mathcal{P}_J(\Phi \, \mathcal{U}^{[t,t]} up)$: conditional instantaneous availability at time $t$
- $\mathcal{P}_J(\mathcal{G}^{[t,t']} up)$: interval availability

We can even nest $\mathcal{P}$ and $\mathcal{S}$:

- $\mathcal{P}_{[0,0.01]}(up_2 \vee up_3 \;\; \mathcal{U}^{[0,10]} \; down)$: The probability of going down within 10 time units after having continuously operated with at least two processors is at most 0.01.
- $\mathcal{S}_{[0.9,1.0]}(\mathcal{P}_{[0.8,1.0]}(\mathcal{G}^{[0,10]} \neg down))$: In the long-run, at least 90% of time is spent in states where the probability that the system will not go down within 10 time units is at least 0.8.

# CSL - Reachability

Let us fix a labelled CTMC $\mathcal{C} = (S, \mathsf{R}, \pi_0, L)$, goal states $B \subseteq S$, and interval of time $I \subseteq \mathbb{R}_{\geq 0}$. We write $\mathit{Reach}^I(B) := \mathit{Paths}(\mathcal{F}^{\,I} B)$.

## Reachability

What is the probability $P(\mathit{Reach}^I(B))$?

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## Lemma:

For any labelled CTMC $\mathcal{C} = (S, \mathrm{R}, \pi_0, L)$, let $\mathcal{C}[B]$ be obtained by making states in $B$ absorbing. Then, the reachability probability $P(Reach^I(B))$ does not change.

# CSL - Reachability - Time-bounded

Theorem:
For $\mathcal{C} = (S, \mathsf{R}, \pi_0, L)$ with an absorbing set of states $B$ and $I = [0, t]$:

$$P(Reach^{[0,t]}(B)) \stackrel{1}{=} P(Reach^{[t,t]}(B)) \stackrel{2}{=} \sum_{s' \in B} \pi_t(s').$$

Proof Sketch:

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Proof Sketch:

1. Show that $\sigma \in Reach^{[0,t]}(B)$ iff $\sigma \in Reach^{[t,t]}(B)$.
2. Trivial.

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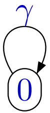
Counterexample:



# CSL - Reachability - Interval Bounded (2)

## Theorem:

Let $\mathcal{C} = (S, \mathrm{R}, \pi_0, L)$ with an absorbing set of states $B$ and $I = [a, b]$ with $a > 0$, then,

$$P(Reach^{[a,b]}(B)) = \sum_{s \in S} \pi_a(s) \cdot P_s(Reach^{[0,b-a]}(B)).$$

## Proof Sketch:

By the theorem of total probability we have:

$$\begin{aligned}
P(Reach^{[a,b]}(B)) &= \sum_{s \in S} P_s(X_a = s) P\left(Reach^{[a,b]}(B) \mid X_a = s'\right) \\
&= \sum_{s \in S} \pi_a(s) P\left(Reach^{[0,b-a]}(B) \mid X_0 = s\right) \\
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\end{aligned}$$

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## Next $\mathcal{X}^{[a,b]} \Phi$:

$$P_s(\mathcal{X}^{[a,b]} \Phi) = (e^{-E(s)a} - e^{-E(s)b}) \sum_{s' \in Sat(\Phi)} \mathrm{E}(s, s').$$

For $I = [0, \infty)$, this simplifies to $P_s(\mathcal{X} \, \Phi) = \sum_{s' \in Sat(\Phi)} \mathrm{E}(s, s')$.

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### Steady State $\mathcal{S}_{<p}(\Phi)$:

Can be reduced to the analogous PCTL model checking problem on the uniformized Markov chain $uni(\mathcal{C})$. Intuition: A CTMC and its uniformized chain have the same steady state distribution.