Quantitative Verification

Chapter 3: Markov chains

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Motivation

Example: Simulation of a die by coins

Knuth & Yao die

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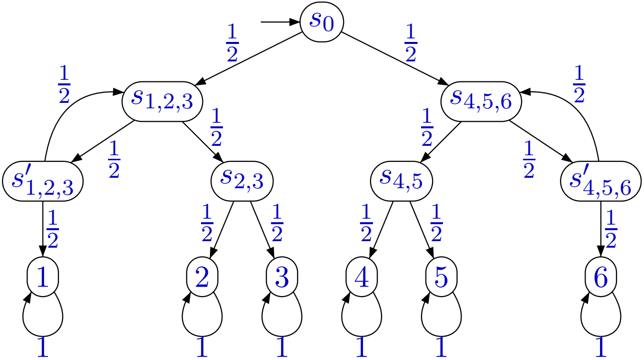
Knuth & Yao die



Question:

- What is the probability of obtaining 2?

DTMC - Graph-based Definition

Definition:

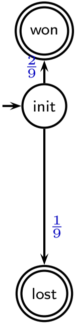
A discrete-time Markov chain (DTMC) is a tuple (S, P, π_0) where

- S is the set of states,
- $\mathrm{P} : S \times S \to [0, 1]$ with $\sum_{s' \in S} \mathrm{P}(s, s') = 1$ is the transitions matrix, and
- $\pi_0 \in [0, 1]^{|S|}$ with $\sum_{s \in S} \pi_0(s) = 1$ is the initial distribution.

Example: Craps

Two dice game:

- First: $\sum \in \{7, 11\} \Rightarrow$ win, $\sum \in \{2, 3, 12\} \Rightarrow$ lose, else $s = \sum$
- Next rolls: $\sum = s \Rightarrow$ win, $\sum = 7 \Rightarrow$ lose, else iterate



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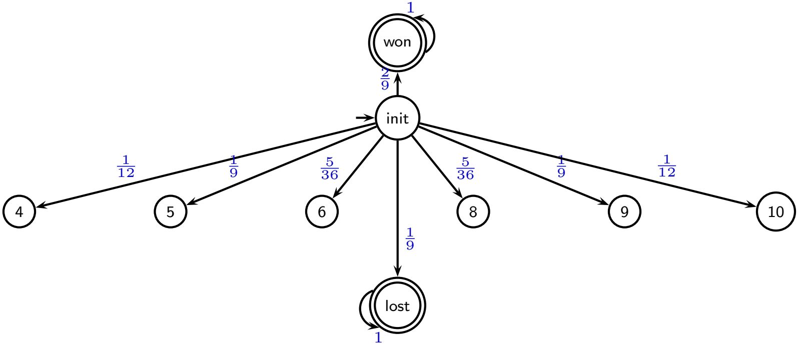

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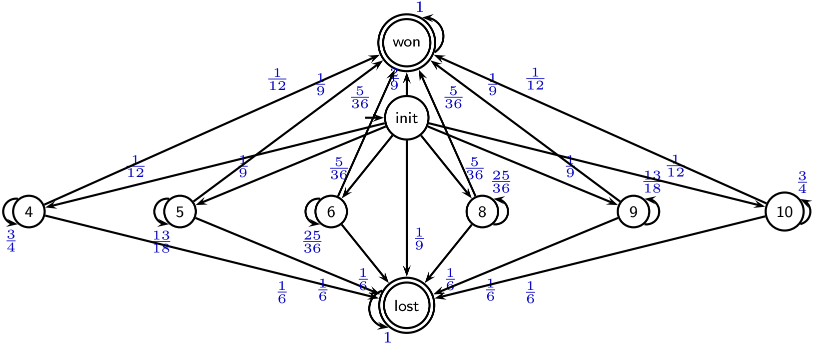


Example: Zero Configuration Networking (Zeroconf)

- Previously: Manual assignment of IP addresses
- Zeroconf: Dynamic configuration of local IPv4 addresses
- Advantage: Simple devices able to communicate automatically

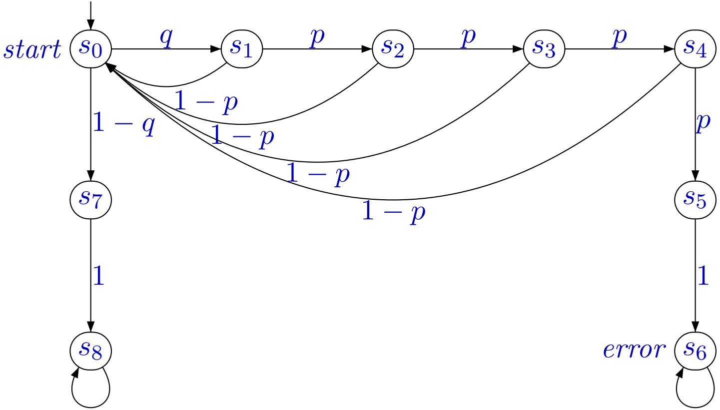
Automatic Private IP Addressing (APIPA) – RFC 3927

- Used when DHCP is configured but unavailable
- Pick randomly an address from 169.254.1.0 – 169.254.254.255
- Find out whether anybody else uses this address (by sending several ARP requests)

Model:

- Randomly pick an address among the K (65024) addresses.
- With m hosts in the network, collision probability is $q = \frac{m}{K}$.
- Send 4 ARP requests.
- In case of collision, the probability of no answer to the ARP request is p (due to the lossy channel)

Example: Zero Configuration Networking (Zeroconf)



For 100 hosts and $p = 0.001$, the probability of error is $\approx 1.55 \cdot 10^{-15}$.

Probabilistic Model Checking

What is probabilistic model checking?

- Probabilistic specifications, e.g. probability of reaching bad states shall be smaller than 0.01.
- Probabilistic model checking is an automatic verification technique for this purpose.

Why quantities?

- Randomized algorithms
- Faults e.g. due to the environment, lossy channels
- Performance analysis, e.g. reliability, availability

Basics of Probability Theory (Recap)

What are probabilities? – Intuition

Throwing a fair coin:

- The outcome head has a probability of 0.5.
- The outcome tail has a probability of 0.5.

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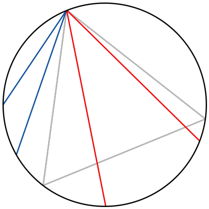
But …[Bertrand's Paradox]

Draw a random chord on the unit circle. What is the probability that its length exceeds the length of a side of the equilateral triangle in the circle?



$\frac{1}{3}$

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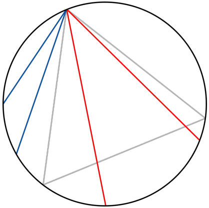
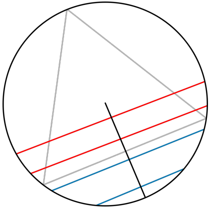


$\frac{1}{3}$



$\frac{1}{2}$

What are probabilities? – Intuition

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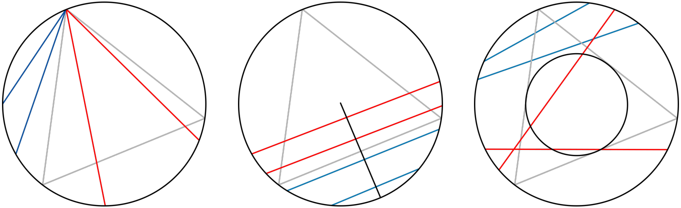

$\frac{1}{3}$ $\frac{1}{2}$ $\frac{1}{4}$

Probability Theory - Probability Space

Definition: Probability Function

Given sample space Ω and σ-algebra $\mathcal{F}$, a probability function $P : \mathcal{F} \to [0, 1]$ satisfies:

- $P(A) \geq 0$ for $A \in \mathcal{F}$,
- $P(\Omega) = 1$, and
- $P(\dot{\bigcup}_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for pairwise disjoint $A_i \in \mathcal{F}$

Definition: Probability Space

A probability space is a tuple $(\Omega, \mathcal{F}, P)$ with a sample space Ω, σ-algebra $\mathcal{F} \subseteq 2^{\Omega}$ and probability function P.

Example

A random real number taken uniformly from the interval $[0, 1]$.

- Sample space: $\Omega = [0, 1]$.

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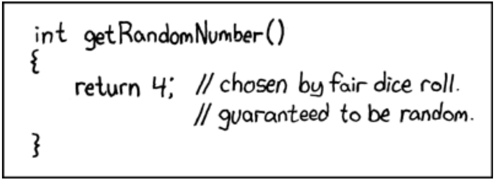
Example

A random real number taken uniformly from the interval $[0,1]$.

- Sample space: $\Omega = [0,1]$.
- σ-algebra: $\mathcal{F}$ is the minimal superset of $\{[a,b] \mid 0 \leq a \leq b \leq 1\}$ closed under complementation and countable union.
- Probability function: $P([a,b]) = (b-a)$, by Carathéodory's extension theorem there is a unique way how to extend it to all elements of $\mathcal{F}$.

Random Variables

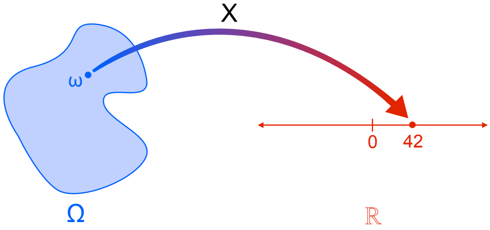


Random Variables - Introduction

Definition: Random Variable

A random variable X is a measurable function $X : \Omega \to I$ to some I.

Elements of I are called random elements. Often $I = \mathbb{R}$:



Example (Bernoulli Trials)

Throwing a coin 3 times: $\Omega_3 = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$.
We define 3 random variables $X_i : \Omega \to \{h, t\}$. For all $x, y, z \in \{h, t\}$,

- $X_1(xyz) = x$,
- $X_2(xyz) = y$,
- $X_3(xyz) = z$.

Stochastic Processes and Markov Chains

Stochastic Processes - Definition

Definition:
Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process is a family of random variables

$$\{X_t \mid t \in T\}$$

defined on $(\Omega, \mathcal{F}, P)$. For each X_t we assume

$$X_t : \Omega \to S$$

where $S = \{s_1, s_2, \dots\}$ is a finite or countable set called state space.

A stochastic process $\{X_t \mid t \in T\}$ is called

- discrete-time if $T = \mathbb{N}$ or
- continuous-time if $T = \mathbb{R}_{\geq 0}$.

For the following lectures we focus on discrete time.

Discrete-time Stochastic Processes – Construction (1)

Example: Weather Forecast

- $S = \{sun, rain\}$,
- we model time as discrete – a random variable for each day:
  - X_0 is the weather today,
  - X_i is the weather in i days.
- how can we set up the probability space to measure e.g. $P(X_i = sun)$?

Discrete-time Stochastic Processes - Construction (2)

Let us fix a state space S. How can we construct the probability space $(\Omega, \mathcal{F}, P)$?

Definition: Sample Space Ω

We define $\Omega = S^\infty$. Then, each X_n maps a sample $\omega = \omega_0\omega_1\ldots$ onto the respective state at time n, i.e.,

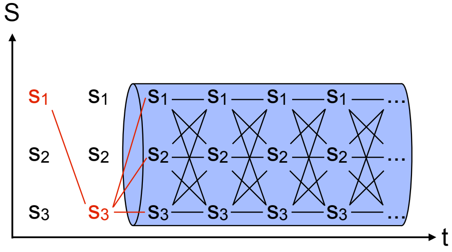
$$(X_n)(\omega) = \omega_n \in S.$$

Discrete-time Stochastic Processes - Construction (3)

Definition: Cylinder Set

For $s_0 \cdots s_n \in S^{n+1}$, we set the cylinder $C(s_0 \ldots s_n) = \{s_0 \cdots s_n \, \omega \in \Omega\}$.

Example:
$S = \{s_1, s_2, s_3\}$ and $C(s_1 s_3)$



Definition: σ-algebra $\mathcal{F}$

We define $\mathcal{F}$ to be the smallest σ-Algebra that contains all cylinder sets, i.e.,

$$\{C(s_0 \ldots s_n) \mid n \in \mathbb{N}, s_i \in S\} \subseteq \mathcal{F}.$$

Check: Is each X_i measurable?
(on the discrete set S we assume the full σ-algebra 2^S).

Discrete-time Stochastic Processes - Construction (4)

How to specify the probability Function P?

We only need to specify it for each $s_0 \cdots s_n \in S^n$

$$P(C(s_0 \ldots s_n)).$$

This amounts to specifying

1. $P(C(s_0))$ for each $s_0 \in S$, and
2. $P(C(s_0 \ldots s_i) \mid C(s_0 \ldots s_{i-1}))$ for each $s_0 \cdots s_i \in S^i$

since

$$\begin{aligned} P(C(s_0 \ldots s_n)) &= P(C(s_0 \ldots s_n) \mid C(s_0 \ldots s_{n-1})) \cdot P(C(s_0 \ldots s_{n-1})) \\ &= P(C(s_0)) \cdot \prod_{i=1}^{n} P(C(s_0 \ldots s_i) \mid C(s_0 \ldots s_{i-1})) \end{aligned}$$

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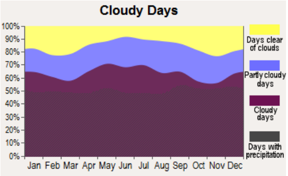
Still, lots of possibilities...

Discrete-time Stochastic Processes – Construction (5)

Weather Example: Option 1 – statistics of days of a year

- the forecast starts on Jan 01,
- a distribution p_j over $\{sun, rain\}$ for each $1 \leq j \leq 365$,
- for each $i \in \mathbb{N}$ and $s_0 \cdots s_i \in S^{i+1}$

$$P(C(s_0 \ldots s_i) \mid C(s_0 \ldots s_{i-1})) = p_{i\,\%\,365}(s_i)$$



Weather Example: Option 2 – two past days

- a distribution $p_{s's''}$ over $\{sun, rain\}$ for each $s', s'' \in S$,
- for each $i \geq 2$ and $s_0 \cdots s_i \in S^{i+1}$

$$P(C(s_0 \ldots s_i) \mid C(s_0 \ldots s_{i-1})) = p_{s_{i-2}s_{i-1}}(s_i)$$

Discrete-time Stochastic Processes - Construction (5)

Weather Example: Option 1 - statistics of days of a year

Not time-homogeneous.

- the forecast starts on Jan 01,
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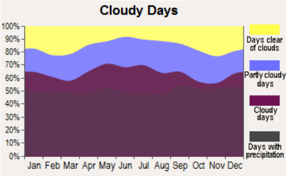

Weather Example: Option 2 - two past days

Not Markovian.

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Here: time-homogeneous Markovian stochastic processes

Stochastic Processes - Restrictions

Definition: Markov

A discrete-time stochastic process $\{X_n \mid n \in \mathbb{N}\}$ is Markov if

$$P(X_n = s_n \mid X_{n-1} = s_{n-1}, \ldots, X_0 = s_0) = P(X_n = s_n \mid X_{n-1} = s_{n-1})$$

for all $n > 1$ and $s_0, \ldots, s_n \in S$ with $P(X_{n-1} = s_{n-1}) > 0$.

Definition: Time-homogeneous

A discrete-time Markov process $\{X_n \mid n \in \mathbb{N}\}$ is time-homogeneous if

$$P(X_{n+1} = s' \mid X_n = s) = P(X_1 = s' \mid X_0 = s)$$

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for all $n > 1$ and $s, s' \in S$ with $P(X_0 = s) > 0$.

А. А. Марков (1886).

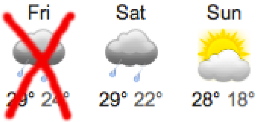
Discrete-time Stochastic Processes - Construction (6)

Weather Example: Option 3 - one past day

- a distribution $p_{s'}$ over $\{sun, rain\}$ for each $s' \in S$,
- for each $i \geq 1$ and $s_0 \cdots s_i \in S^{i+1}$

  $$P(C(s_0 \ldots s_i) \mid C(s_0 \ldots s_{i-1})) = p_{s_{i-1}}(s_i)$$

- a distribution π over $\{sun, rain\}$ such that $P(C(s_0)) = \pi(s_0)$.



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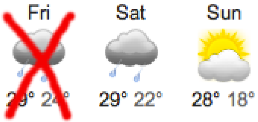

Overly restrictive, isn't it?

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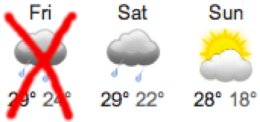


Overly restrictive, isn't it?

Not really – one only needs to extend the state space

- $S = \{1, \ldots, 365\} \times \{sun, rain\} \times \{sun, rain\}$,
- now each state encodes current day of the year, current weather, and weather yesterday,
- we can define over S a time-homogeneous Markov process based on both Options 1 & 2 given earlier.

Discrete-time Markov Chains
DTMC

DTMC - Relation of Definitions

Stochastic process $\rightarrow$ Graph based

Given a discrete-time homogeneous Markov process $\{X(n) \mid n \in \mathbb{N}\}$

- with state space S,
- defined on a probability space $(\Omega, \mathcal{F}, P)$

we take over the state space S and define

- $\mathrm{P}(s, s') = P(X_n = s' \mid X_{n-1} = s)$ for an arbitrary $n \in \mathbb{N}$ and
- $\pi_0(s) = P(X_0 = s)$.

Graph based $\rightarrow$ stochastic process

Given a DTMC (S, P, π_0), we set Ω to S^∞, $\mathcal{F}$ to the smallest σ-Algebra containing all cylinder sets and

$$P(C(s_0 \dots s_n)) = \pi_0(s_0) \cdot \prod_{1 \leq i \leq n} \mathrm{P}(s_{i-1}, s_i)$$

which uniquely defines the probability function P on $\mathcal{F}$.

DTMC - Conditional Probability and Expectation

Let (S, P, π_0) be a DTMC. We denote by

- P_s the probability function of DTMC $(S, \mathsf{P}, \delta_s)$ where

$$\delta_s(s') = \begin{cases} 1 & \text{if } s' = s \\ 0 & \text{otherwise} \end{cases}$$

- E_s the expectation with respect to P_s

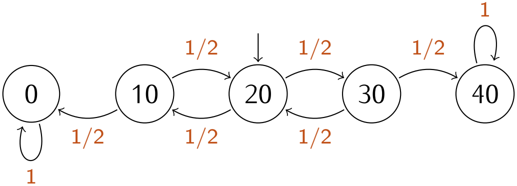
Analysis questions

- Transient analysis
- Steady-state analysis
- Rewards
- Reachability
- Probabilistic logics

DTMC - Transient Analysis

DTMC - Transient Analysis - Example (1)

Example: Gambling with a Limit



What is the probability of being in state 0 after 3 steps?

DTMC - Transient Analysis - n-step Probabilities

Definition:

Given a DTMC (S, P, π_0), we assume w.l.o.g. $S = \{0, 1, \dots\}$ and write $p_{ij} = \mathrm{P}(i, j)$. Further, we have

- $\mathrm{P}^{(1)} = \mathrm{P} = (p_{ij})$ is the 1-step transition matrix
- $\mathrm{P}^{(n)} = (p_{ij}^{(n)})$ denotes the n-step transition matrix with

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \quad (= P(X_{k+n} = j \mid X_k = i)).$$

How can we compute these probabilities?

Definition: Chapman-Kolmogorov Equation

Application of the law of total probability to the n-step transition probabilities $p_{ij}^{(n)}$ results in the Chapman-Kolmogorov Equation

$$p_{ij}^{(n)} = \sum_{h \in S} p_{ih}^{(m)} p_{hj}^{(n-m)} \qquad \forall 0 < m < n.$$

Consequently, we have $\mathrm{P}^{(n)} = \mathrm{P}\mathrm{P}^{(n-1)} = \cdots = \mathrm{P}^{n}$.

DTMC - Transient Analysis - Chapman-Kolmogorov

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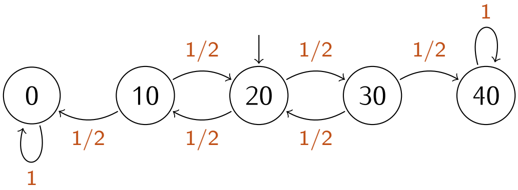
Definition: Transient Probability Distribution

The transient probability distribution at time $n > 0$ is defined by

$$\pi_n = \pi_{n-1}\mathsf{P} = \pi_0 \mathsf{P}^n.$$

DTMC - Transient Analysis - Example (2)

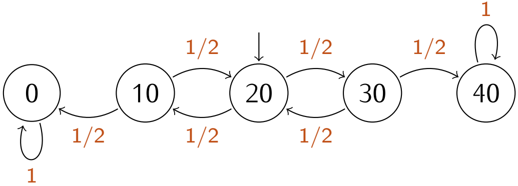


Example:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.25 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- For $\pi_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $\pi_2 = \pi_0 P^2 = \begin{bmatrix} 0.25 & 0 & 0.5 & 0 & 0.25 \end{bmatrix}$.
- For, $\pi_0 = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{bmatrix}$, $\pi_2 = \pi_0 P^2 = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{bmatrix}$.
  Actually, $\pi_n = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{bmatrix}$ for all $n \in \mathbb{N}$!

DTMC - Transient Analysis - Example (2)



Example:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.25 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- For $\pi_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $\pi_2 = \pi_0 P^2 = \begin{bmatrix} 0.25 & 0 & 0.5 & 0 & 0.25 \end{bmatrix}$.
- For, $\pi_0 = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{bmatrix}$, $\pi_2 = \pi_0 P^2 = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{bmatrix}$. Actually, $\pi_n = \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0.6 \end{bmatrix}$ for all $n \in \mathbb{N}$!

Are there other "stable" distributions?

DTMC - Steady State Analysis

DTMC - Steady State Analysis - Definitions

Definition: Stationary Distribution

A distribution π is stationary if

$$\pi = \pi \mathsf{P}.$$

Stationary distribution is generally not unique.

DTMC - Steady State Analysis - Definitions

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Stationary distribution is generally not unique.

Definition: Limiting Distribution

$$\pi^* := \lim_{n \to \infty} \pi_n = \lim_{n \to \infty} \pi_0 \mathsf{P}^n = \pi_0 \lim_{n \to \infty} P^n = \pi_0 \mathsf{P}^*.$$

The limit can depend on π_0 and does not need to exist.

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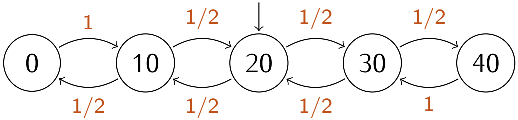
Connection between stationary and limiting?

DTMC - Steady-State Analysis - Periodicity

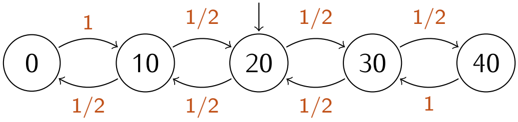
Example: Gambling with Social Guarantees



What are the stationary and limiting distributions?

DTMC - Steady-State Analysis - Periodicity

Example: Gambling with Social Guarantees



What are the stationary and limiting distributions?

Definition: Periodicity

The period of a state i is defined as

$$d_i = \mathrm{gcd}\{n \mid p_{ii}^n > 0\}.$$

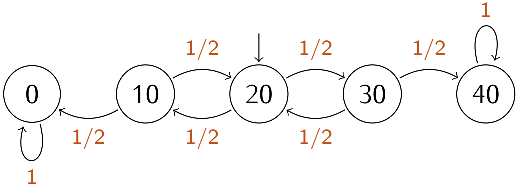
A state i is called aperiodic if $d_i = 1$ and periodic with period d_i otherwise. A Markov chain is aperiodic if all states are aperiodic.

Lemma

In a finite aperiodic Markov chain, the limiting distribution exists.

DTMC - Steady-State Analysis - Irreducibility (1)

Example



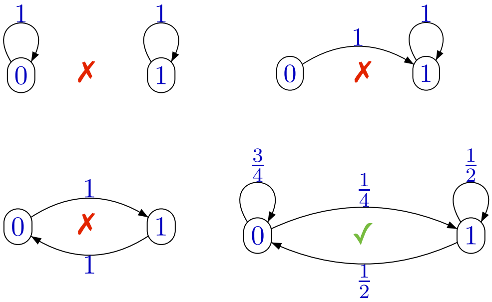
DTMC - Steady-State Analysis - Irreducibility (2)

Definition:

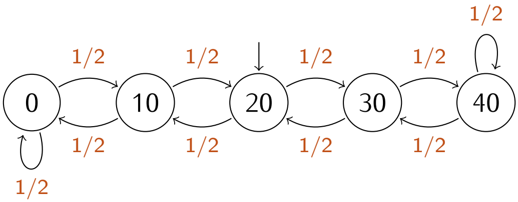
A DTMC is called irreducible if for all states $i, j \in S$ we have $p_{ij}^n > 0$ for some $n \geq 1$.

Lemma

In an aperiodic and irreducible Markov chain, the limiting distribution exists and does not depend on π_0.

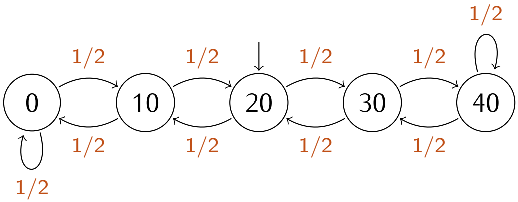


DTMC - Steady-State Analysis - Irreducibility (3)



What is the stationary / limiting distribution?

DTMC - Steady-State Analysis - Irreducibility (3)



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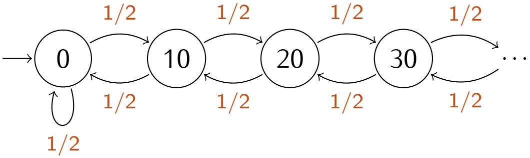

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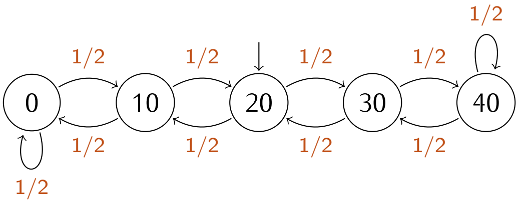
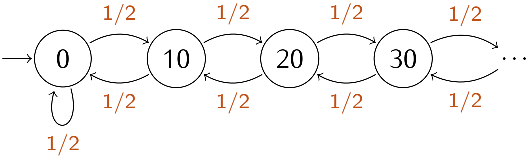


What is the stationary / limiting distribution?



Lemma

In a finite aperiodic and irreducible Markov chain, the limiting distribution exists, does not depend on π_0, and equals the unique stationary distribution.

DTMC - Steady-State Analysis - Recurrence (1)

Definition:
Let $f_{ij}^{(n)} = P(X_n = j \wedge \forall 1 \leq k < n : X_k \neq j \mid X_0 = i)$ for $n \geq 1$ be the n-step hitting probability. The hitting probability is defined as

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

and a state i is called

- transient if $f_{ii} < 1$ and
- recurrent if $f_{ii} = 1$.

DTMC - Steady-State Analysis - Recurrence (2)

Definition:

Denoting expectation $m_{ij} = \sum_{n=1}^{\infty} n \cdot f_{ij}^{(n)}$, a recurrent state i is called

- positive recurrent or recurrent non-null if $m_{ii} < \infty$ and
- recurrent null if $m_{ii} = \infty$.

Lemma

The states of an irreducible DTMC are all of the same type, i.e.,

- *all periodic or*
- *all aperiodic and transient or*
- *all aperiodic and recurrent null or*
- *all aperiodic and recurrent non-null.*

Definition: Ergodicity

A DTMC is ergodic if all its states are irreducible, aperiodic and recurrent non-null.

Theorem

In an ergodic Markov chain, the limiting distribution exists, does not depend on π_0, and equals the unique stationary distribution.

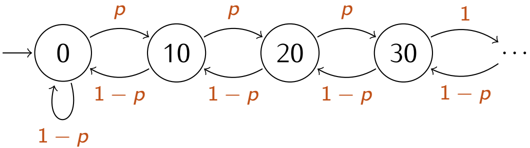
As a consequence, the steady-state distribution can be computed by solving the equation system

$$\pi = \pi \mathsf{P}, \sum_{x \in S} \pi_s = 1.$$

Note: The Lemma for finite DTMC follows from the theorem as every irreducible finite DTMC is positive recurrent.

DTMC - Steady-State Analysis - Ergodicity

Example: Unbounded Gambling with House Edge



The DTMC is only ergodic for $p \in [0, 0.5)$.

DTMC - Rewards

DTMC - Rewards - Definitions

Definition

A reward Markov chain is a tuple $(S, \mathsf{P}, \pi_0, r)$ where (S, P, π_0) is a Markov chain and $r : S \to \mathbb{Z}$ is a reward function.

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Every run $\rho = s_0, s_1, \ldots$ induces a sequence of values $r(s_0), r(s_1), \ldots$

Value of the whole run can be defined as

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$\sum_{i=0}^{\infty} \lambda^i r(s_i)$ for some $0 < \lambda < 1$

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Definition

The expected average reward is

$$EAR := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \mathbb{E}[r(X_i)]$$

DTMC - Rewards - Solution Sketch

Definition: Time-average Distribution

$$\hat{\pi} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} \pi_i .$$

$\hat{\pi}(s)$ expresses the ratio of time spent in s on the long run.

[1] More details later for Markov decision processes.

DTMC - Rewards - Solution Sketch

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$\hat{\pi}(s)$ expresses the ratio of time spent in s on the long run.

Lemma

1. $\mathbb{E}[r(X_i)] = \sum_{s \in S} \pi_i(s) \cdot r(s)$.
2. If $\hat{\pi}$ exists then $EAR = \sum_{s \in S} \hat{\pi}(s) \cdot r(s)$.
3. If limiting distribution exists, it coincides with $\hat{\pi}$.

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Algortithm

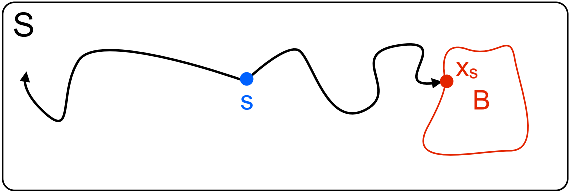
1. Compute $\hat{\pi}$ (or limiting distribution if possible).[1]
2. Return $\sum_{s\in S} \hat{\pi}(s) \cdot r(s)$.

[1] More details later for Markov decision processes.

DTMC - Reachability

Definition: Reachability

Given a DTMC (S, P, π_0), what is the probability of eventually reaching a set of goal states $B \subseteq S$?



Let $x(s)$ denote $P_s(\Diamond B)$ where $\Diamond B = \{s_0 s_1 \cdots \mid \exists i : s_i \in B\}$. Then

- $s \in B$: $\quad x(s) =$
- $s \in S \setminus B :$ $\quad x(s) =$

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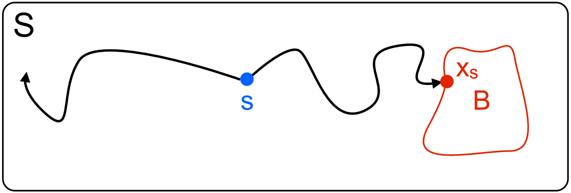


Let $x(s)$ denote $P_s(\Diamond B)$ where $\Diamond B = \{s_0 s_1 \cdots \mid \exists i : s_i \in B\}$. Then

- $s \in B$: $\quad x(s) = 1$
- $s \in S \setminus B$: $\quad x(s) = \sum_{t \in S \setminus B} \mathrm{P}(s,t)x(t) + \sum_{u \in B} \mathrm{P}(s,u)$.

DTMC - Reachability

Lemma (Reachability Matrix Form)

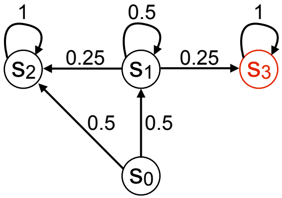
Given a DTMC (S, P, π_0)*, the column vector* $\mathsf{x} = (x(s))_{s \in S \setminus B}$ *of probabilities* $x(s) = P_s(\lozenge B)$ *satisfies the constraint*

$$\mathsf{x} = \mathsf{A}\mathsf{x} + \mathsf{b},$$

where matrix A *is the submatrix of* P *for states* $S \setminus B$ *and* $\mathsf{b} = (b(s))_{s \in S \setminus B}$ *is the column vector with* $b(s) = \sum_{u \in B} \mathsf{P}(s, u)$.

DTMC - Reachability

Example:

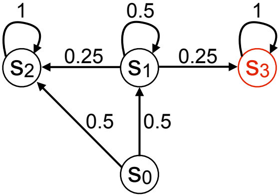


$$P = \begin{array}{cccc} & A & & b \\ \left[\begin{array}{ccc} 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.25 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right. & & & \left.\begin{array}{c} 0 \\ 0.25 \\ 0 \\ 1 \end{array}\right] \end{array}$$

$B = \{s_3\}$

The vector $x = \begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} 0.25 & 0.5 & 0 \end{bmatrix}^T$ satisfies the equation system $x = Ax + b$.

Example:



$$P = \begin{array}{c} \begin{array}{cccc} & A & & b \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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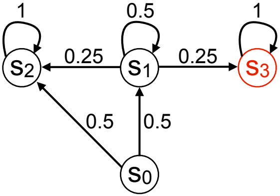
Is it the only solution?

Example:



$$\begin{array}{cc} & \begin{array}{ccc|c} \quad & A & \quad & b \end{array} \\ P = & \left[\begin{array}{ccc|c} 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

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Is it the only solution?

- No! Consider, e.g., $\begin{bmatrix} 0.55 & 0.7 & 0.4 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

What is the equation system for these probabilities?

DTMC - Reachability - Solution

Let $S_{=0} = \{s \mid P_s(\Diamond\ B) = 0\}$ and $S_? = S \setminus (S_{=0} \cup B)$.
Let $\Diamond^{\leq n}\ B = \{s_0 s_1 \cdots \mid \exists i \leq n : s_i \in B\}$ be the set of runs reaching B from state s within n steps.

DTMC - Reachability - Solution

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Let $\Diamond^{\leq n}\ B = \{s_0 s_1 \cdots \mid \exists i \leq n : s_i \in B\}$ be the set of runs reaching B from state s within n steps.

Theorem:

The column vector $\mathrm{x} = (x(s))_{s \in S_?}$ of probabilities $x(s) = P_s(\Diamond\ B)$ is the unique solution of the equation system

$$\mathrm{x} = \mathrm{A}\mathrm{x} + \mathrm{b},$$

where $\mathrm{A} = (\mathrm{P}(s,t))_{s,t \in S_?}$, $\mathrm{b} = (b(s))_{s \in S_?}$ with $b(s) = \sum_{u \in B} \mathrm{P}(s,u)$.

Furthermore, for $\mathrm{x}_0 = (0)_{s \in S_?}$ and $\mathrm{x}_i = \mathrm{A}\mathrm{x}_{i-1} + \mathrm{b}$ for any $i \geq 1$,

1. $\mathrm{x}_n(s) = P_s(\Diamond^{\leq n}\ B)$ for $s \in S_?$,
2. x_i is increasing, and
3. $\mathrm{x} = \lim_{n \to \infty} \mathrm{x}_n$.

DTMC - Conditional Reachability - Proof

Proof Sketch:

- $(x_s)_{x \in S_?}$ is a solution: by inserting into definition.
- Unique solution: By contradiction. Assume y is another solution, then $\mathsf{x} - \mathsf{y} = \mathsf{A}(\mathsf{x} - \mathsf{y})$. One can show that $\mathsf{A} - \mathsf{I}$ is invertible, thus $(\mathsf{A} - \mathsf{I})(\mathsf{x} - \mathsf{y}) = 0$ yields $\mathsf{x} - \mathsf{y} = (\mathsf{A} - \mathsf{I})^{-1}0 = 0$ and finally $\mathsf{x} = \mathsf{y}$ [2].

Furthermore,

1. From the definitions, by straightforward induction.
2. From 1. since $\lozenge^{\leq n} B \subseteq \lozenge^{\leq n+1} B$.
3. Since $\lozenge B = \bigcup_{n \in \mathbb{N}} \lozenge^{\leq n} B$.

□

[2] cf. page 766 of Principles of Model Checking

Algorithmic aspects

Algorithmic Aspects - Summary of Equation Systems

Equation Systems

- Transient analysis: $\pi_n = \pi_0 \mathrm{P}^n = \pi_{n-1}\mathrm{P}$
- Steady-state analysis: $\pi \mathrm{P} = \pi, \pi \cdot \mathbf{1} = \sum_{s \in S} \pi(s) = 1$ (ergodic)
- Reachability: $\mathrm{x} = \mathrm{A}\mathrm{x} + \mathrm{b}$ (with $(x(s))_{s \in S_?}$)

Solution Techniques

1. Analytic solution, e.g. by Gaussian elimination
2. Iterative power method ($\pi_n \to \pi$ and $\mathrm{x}_n \to \mathrm{x}$ for $n \to \infty$)
3. Iterative methods for solving large systems of linear equations, e.g. Jacobi, Gauss-Seidel

Missing pieces

a. finding out whether a DTMC is ergodic,
b. computing $S_? = S \setminus \{s \mid P_s(\Diamond B) = 0\}$,
c. efficient representation of P.

Algorithmic Aspects: a. Ergodicity of finite DTMC (1)

Ergodicity = Irreducibility + Aperidocity + P. Recurrence

- A DTMC is called irreducible if for all states $i, j \in S$ we have $p_{ij}^n > 0$ for some $n \geq 1$.
- A state i is called aperiodic if $\gcd\{n \mid p_{ii}^n > 0\} = 1$.
- A state i is called positive recurrent if $f_{ii} = 1$ and $m_{ii} < \infty$.

How do we tell that a finite DTMC is ergodic?

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How do we tell that a finite DTMC is ergodic?

By analysis of the induced graph!

For a DTMC $(S, \mathrm{P}, \pi(0))$ we define the induced directed graph (S, E) with $E = \{(s, s') \mid \mathrm{P}(s, s') > 0\}$.

Recall:

- A directed graph is called strongly connected if there is a path from each vertex to every other vertex.
- Strongly connected components (SCC) are its maximal strongly connected subgraphs.
- A SCC T is bottom (BSCC) if no $s \notin T$ is reachable from T.

Algorithmic Aspects: a. Ergodicity of finite DTMC (2)

Ergodicity = Irreducibility + Aperidocity + P. Recurrence

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Theorem:

For finite DTMCs, it holds that:

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Theorem:

For finite DTMCs, it holds that:

- The DTMC is irreducible iff the induced graph is strongly connected.

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Theorem:

For finite DTMCs, it holds that:

- The DTMC is irreducible iff the induced graph is strongly connected.
- A state in a BSCC is aperiodic iff the BSCC is aperiodic, i.e. the greatest common divisor of the lengths of all its cycles is 1.

Algorithmic Aspects: a. Ergodicity of finite DTMC (2)

Ergodicity = Irreducibility + Aperidocity + P. Recurrence

- A DTMC is called irreducible if for all states $i, j \in S$ we have $p^n_{ij} > 0$ for some $n \geq 1$.
- A state i is called aperiodic if $\gcd\{n \mid p^n_{ii} > 0\} = 1$.
- A state i is called positive recurrent if $f_{ii} = 1$ and $m_{ii} < \infty$.

Theorem:

For finite DTMCs, it holds that:

- The DTMC is irreducible iff the induced graph is strongly connected.
- A state in a BSCC is aperiodic iff the BSCC is aperiodic, i.e. the greatest common divisor of the lengths of all its cycles is 1.
- A state is positive recurrent iff it belongs to a BSCC otherwise it is transient.

Algorithmic Aspects: a. Ergodicity of finite DTMC (3)

How to check: is gcd of the lengths of all cycles of a strongly connected graph 1?

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How to check: is gcd of the lengths of all cycles of a strongly connected graph 1?

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- in time $\mathcal{O}(n+m)$?

Algorithmic Aspects: a. Ergodicity of finite DTMC (3)

How to check: is gcd of the lengths of all cycles of a strongly connected graph 1?

- $\gcd\{n \geq 1 \mid \exists s : \mathrm{P}^n(s,s) > 0\} = 1$
- in time $\mathcal{O}(n+m)$? By the following DFS-based procedure:

Algorithm: PERIOD(vertex v, unsigned $level$: init 0)

global $period$: init 0;
if $period = 1$ **then**
    return
end
if v is unmarked **then**
    mark v;
    $v_{level} = level$;
    for $v' \in out(v)$ **do**
        PERIOD(v',$level + 1$)
    end
else
    $period = \gcd(period, level - v_{level})$;
end

Algorithmic Aspects: b. Computing the set $S_?$

We have $S_? = S \setminus (B \cup S_{=0})$ where $S_{=0} = \{s \mid P_s(\Diamond B) = 0\}$.
Hence,

$$s \in S_{=0} \quad \text{iff} \qquad p^n_{ss'} = 0 \quad \text{for all } n \geq 1 \text{ and } s' \in B.$$

Algorithmic Aspects: b. Computing the set $S_?$

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Hence,

$$s \in S_{=0} \quad \text{iff} \qquad p^n_{ss'} = 0 \quad \text{for all } n \geq 1 \text{ and } s' \in B.$$

This can be again easily checked from the induced graph:

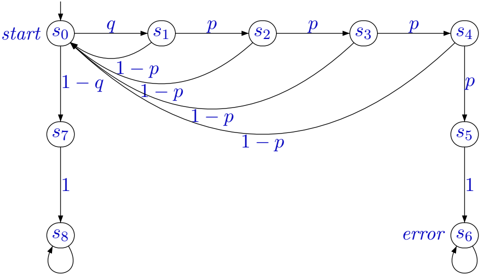
Lemma

We have $s \in S_{=0}$ iff there is no path from s to any state from B.

Proof.

Easy from the fact that $p^n_{ss'} > 0$ iff there is a path of length n to s'. $\square$

Algorithmic Aspects: c. Efficient Representations



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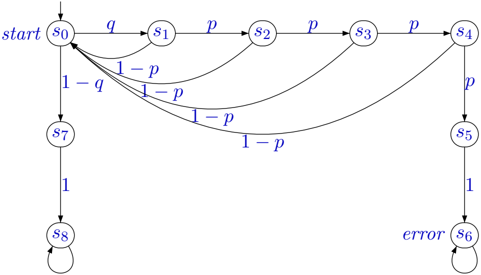
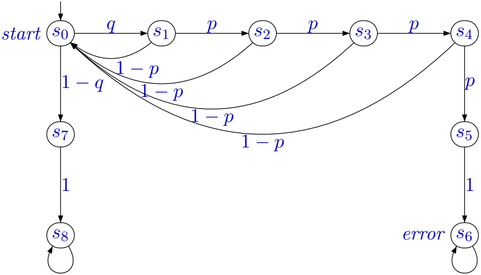


1. There are many 0 entries in the transition matrix.

Sparse matrices offer a more concise storage.

Algorithmic Aspects: c. Efficient Representations



1. There are many 0 entries in the transition matrix.

Sparse matrices offer a more concise storage.

2. There are many similar entries in the transition matrix.

Multi-terminal binary decision diagrams offer a more concise storage, using automata theory.

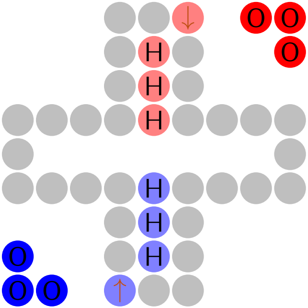
DTMC - Probabilistic Temporal Logics for Specifying Complex Properties

Logics - Adding Labels to DTMC

Definition:
A labeled DTMC is a tuple $\mathcal{D} = (S, \mathrm{P}, \pi_0, L)$ with $L : S \to 2^{AP}$, where

- AP is a set of atomic propositions and
- L is a labeling function, where $L(s)$ specifies which properties hold in state $s \in S$.

Logics - Examples of Properties



States and transitions

state = configuration of the game;
transition = rolling the dice and acting (randomly) based on the result.

State labels

- init, rwin, bwin, rkicked, bkicked, ...
- r30, r21, ...,
- b30, b21,...,

Examples of Properties

- the game cannot return back to start
- at any time, the game eventually ends with prob. 1
- at any time, the game ends within 100 dice rolls with prob. ≥ 0.5
- the probability of winning without ever being kicked out is ≤ 0.3

How to specify them formally?

Logics - Temporal Logics - non-probabilistic (1)

Linear-time view

- corresponds to our (human) perception of time
- can specify properties of one concrete linear execution of the system

Example: eventually red player is kicked out followed immediately by blue player being kicked out.

Branching-time view

- views future as a set of all possibilities
- can specify properties of all executions from a given state – specifies execution trees

Example: in every computation it is always possible to return to the initial state.

Logics - Temporal Logics - non-probabilistic (2)

Linear Temporal Logic (LTL)

Syntax for formulae specifying executions:

$$\psi = \mathit{true} \mid a \mid \psi \wedge \psi \mid \neg\psi \mid \mathcal{X}\ \psi \mid \psi\ \mathcal{U}\ \psi \mid \mathcal{F}\ \psi \mid \mathcal{G}\ \psi$$

Example: eventually red player is kicked out followed immediately by blue player being kicked out: $\mathcal{F}\ (\mathit{rkicked} \wedge \mathcal{X}\ \mathit{bkicked})$

Question: do all executions satisfy the given LTL formula?

Computation Tree Logic (CTL)

Syntax for specifying states:

$$\phi = \mathit{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid A\ \psi \mid E\ \psi$$

Syntax for specifying executions:

$$\psi = \mathcal{X}\ \phi \mid \phi\ \mathcal{U}\ \phi \mid \mathcal{F}\ \phi \mid \mathcal{G}\ \phi$$

Example: in all computations it is always possible to return to initial state: $A\ \mathcal{G}\ \ E\ \mathcal{F}\ \mathit{init}$

Question: does the given state satisfy the given CTL state formula?

Logics - LTL

Syntax

$\psi = \mathit{true} \mid a \mid \psi \wedge \psi \mid \neg\psi \mid \mathcal{X}\ \psi \mid \psi\ \mathcal{U}\ \psi.$

Semantics (for a path $\omega = s_0 s_1 \cdots$)

- $\omega \models \mathit{true}$ (always),
- $\omega \models a$ iff $a \in L(s_0)$,
- $\omega \models \psi_1 \wedge \psi_2$ iff $\omega \models \psi_1$ and $\omega \models \psi_2$,
- $\omega \models \neg\psi$ iff $\omega \not\models \psi$,
- $\omega \models \mathcal{X}\ \psi$ iff $s_1 s_2 \cdots \models \psi$,

	Ψ										

- $\omega \models \psi_1\ \mathcal{U}\ \psi_2$ iff $\exists i \geq 0 : s_i s_{i+1} \cdots \models \psi_2$ and $\forall j < i : s_j s_{j+1} \cdots \models \psi_1$.

Ψ_1	$\cdot$	$\cdot$	$\cdot$	Ψ_1	Ψ_2						

Syntactic sugar

- $\mathcal{F}\ \psi \equiv$
- $\mathcal{G}\ \psi \equiv$

Logics - LTL

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Ψ_1	$\cdot$	$\cdot$	$\cdot$	Ψ_1	Ψ_2						

Syntactic sugar

- $\mathcal{F}\ \psi \equiv \mathit{true}\ \mathcal{U}\ \psi$
- $\mathcal{G}\ \psi \equiv \neg(\mathit{true}\ \mathcal{U}\ \neg\psi) \quad (\equiv \neg\mathcal{F}\ \neg\psi)$

Logics - CTL

Syntax

State formulae:

$$\phi = \mathit{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid A\ \psi \mid E\ \psi$$

where ψ is a path formula.

Path formulae:

$$\psi = \mathcal{X}\ \phi \mid \phi\ \mathcal{U}\ \phi$$

where ϕ is a state formula.

Semantics

For a state s:

- $s \models \mathit{true}$ (always),
- $s \models a$ iff $a \in L(s)$,
- $s \models \phi_1 \wedge \phi_2$ iff $s \models \phi_1$ and $s \models \phi_2$,
- $s \models \neg\phi$ iff $s \not\models \phi$,
- $s \models A\psi$ iff $\omega \models \psi$ for all paths $\omega = s_0 s_1 \cdots$ with $s_0 = s$,
- $s \models E\psi$ iff $\omega \models \psi$ for some path $\omega = s_0 s_1 \cdots$ with $s_0 = s$.

For a path $\omega = s_0 s_1 \cdots$:

- $\omega \models \mathcal{X}\ \phi$ iff $s_1 s_2 \cdots$ satisfies ϕ,

	Φ										

- $\omega \models \phi_1\ \mathcal{U}\ \phi_2$ iff $\exists i :$ $s_i s_{i+1} \cdots \models \phi_2$ and $\forall j < i : s_j s_{j+1} \cdots \models \phi_1$.

Φ_1	$\cdot$	$\cdot$	$\cdot$	Φ_1	Φ_2						

Logics - Temporal Logics - non-probabilistic (2)

Linear Temporal Logic (LTL)

Syntax for formulae specifying executions:

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Example: eventually red player is kicked out followed immediately by blue player being kicked out: $\mathcal{F}\ (\mathit{rkicked} \wedge \mathcal{X}\ \mathit{bkicked})$

Question: do all executions satisfy the given LTL formula?

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Syntax for specifying executions:

$$\psi = \mathcal{X}\ \phi \mid \phi\ \mathcal{U}\ \phi \mid \mathcal{F}\ \phi \mid \mathcal{G}\ \phi$$

Example: in all computations it is always possible to return to initial state: $A\ \mathcal{G}\ \ E\ \mathcal{F}\ \mathit{init}$

Question: does the given state satisfy the given CTL state formula?

Logics - Temporal Logics - probabilistic

Linear Temporal Logic (LTL) + probabilities

Syntax for formulae specifying executions:

$$\psi = \mathit{true} \mid a \mid \psi \wedge \psi \mid \neg\psi \mid \mathcal{X}\ \psi \mid \psi\ \mathcal{U}\ \psi \mid \mathcal{F}\ \psi \mid \mathcal{G}\ \psi$$

Example: with prob. ≥ 0.8, eventually red player is kicked out followed immediately by blue player being kicked out:

$$P(\mathcal{F}\ (\mathit{rkicked} \wedge \mathcal{X}\ \mathit{bkicked})) \geq 0.8$$

Question: is the formula satisfied by executions of given probability?

Logics - Temporal Logics - probabilistic

Linear Temporal Logic (LTL) + probabilities

Syntax for formulae specifying executions:

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$$P(\mathcal{F}\ (\mathit{rkicked} \wedge \mathcal{X}\ \mathit{bkicked})) \geq 0.8$$

Question: is the formula satisfied by executions of given probability?

Probabilitic Computation Tree Logic (PCTL)

Syntax for specifying states:

$$\phi = \mathit{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid \mathcal{P}_J\ \psi$$

Syntax for specifying executions:

$$\psi = \mathcal{X}\ \phi \mid \phi\, \mathcal{U}\ \phi \mid \phi\ \mathcal{U}^{\leq k}\phi \mid \mathcal{F}\ \phi \mid \mathcal{G}\ \phi$$

Example: with prob. at least 0.5 the probability to return to initial state is always at least 0.1: $P_{\geq 0.5}\ \mathcal{G}\ \ P_{\geq 0.1}\ \mathcal{F}\ \mathit{init}$

Question: does the given state satisfy the given PCTL state formula?

Logics - PCTL - Examples

Syntactic sugar:

- $\phi_1 \vee \phi_2 \equiv \neg(\neg\phi_1 \wedge \neg\phi_2)$, $\phi_1 \Rightarrow \phi_2 \equiv \neg\phi_1 \vee \phi_2$, etc.
- ≤ 0.5 denotes the interval $[0, 0.5]$, $= 1$ denotes $[1, 1]$, etc.

Examples:

- A fair die:

$$\bigwedge_{i \in \{1,\ldots,6\}} \mathcal{P}_{=\frac{1}{6}}(\mathcal{F}\ i).$$

- The probability of winning "Who wants to be a millionaire" without using any joker should be negligible:

$$\mathcal{P}_{<1e-10}(\neg(J_{50\%} \vee J_{audience} \vee J_{telephone})\ \ \mathcal{U}\ \ win).$$

Logics - PCTL - Semantics

Semantics

For a state s:

- $s \models true$ (always),
- $s \models a$ iff $a \in L(s)$,
- $s \models \phi_1 \wedge \phi_2$ iff $s \models \phi_1$ and $s \models \phi_2$,
- $s \models \neg\phi$ iff $s \not\models \phi$,
- $\boxed{s \models \mathcal{P}_J(\psi) \quad \text{iff } P_s(Paths(\psi)) \in J}$

For a path $\omega = s_0 s_1 \cdots$:

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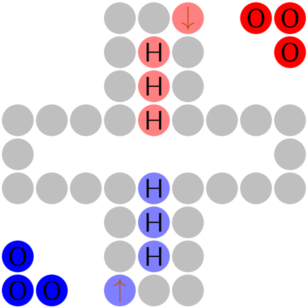
	Φ									

- $\omega \models \phi_1\ \mathcal{U}\ \phi_2$ iff $\exists i :$ $s_i s_{i+1} \cdots \models \phi_2$ and $\forall j < i : s_j s_{j+1} \cdots \models \phi_1$.

Φ_1	·	·	·	·	Φ_1	Φ_2				

- $\omega \models \phi_1\ \mathcal{U}^{\leq n} \phi_2$ iff $\exists i \leq n :$ $s_i s_{i+1} \cdots \models \phi_2$ and $\forall j < i : s_j s_{j+1} \cdots \models \phi_1$.

Φ_1	·	·	·	·	Φ_1	Φ_2				

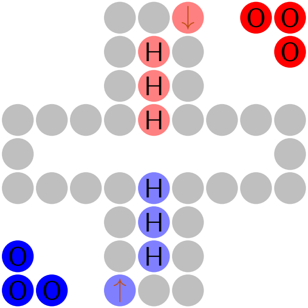


Examples of Properties

1. the game cannot return back to start
2. at any time, the game eventually ends with prob. 1
3. at any time, the game ends within 100 dice rolls with prob. ≥ 0.5
4. the probability of winning without ever being kicked out is ≤ 0.3

Formally

Logics - Examples of Properties



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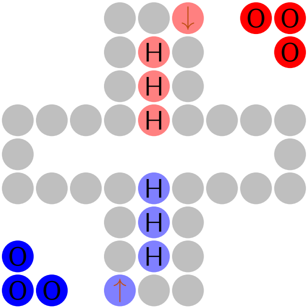
1. $P(\mathcal{X}\ \mathcal{G}\ \neg init) = 1$ (LTL + prob.)
   $P_{=1}(\mathcal{X}\ P_{=0}(\mathcal{G}\ \neg init))$ (PCTL)



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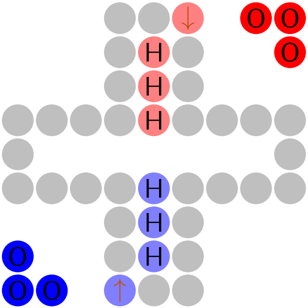
Logics - Examples of Properties



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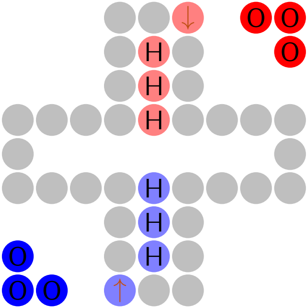
Logics - Examples of Properties



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4. $P((\neg rkicked \wedge \neg bkicked)\ \mathcal{U}\ (rwin \vee bwin)) \leq 0.3$ (LTL + prob.)

PCTL Model Checking Algorithm

PCTL Model Checking

Definition: PCTL Model Checking

Let $\mathcal{D} = (S, \mathrm{P}, \pi_0, L)$ be a DTMC, Φ a PCTL state formula and $s \in S$.
The model checking problem is to decide whether $s \models \Phi$.

Theorem

The PCTL model checking problem can be decided in time polynomial in $|\mathcal{D}|$, linear in $|\Phi|$, and linear in the maximum step bound n.

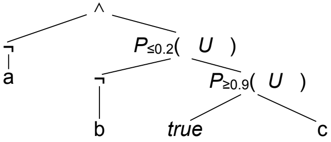
PCTL Model Checking - Algorithm - Outline (1)

Algorithm:

Consider the bottom-up traversal of the parse tree of Φ:

- The leaves are $a \in AP$ or *true* and
- the inner nodes are:
  - unary – labelled with the operator $\neg$ or $\mathcal{P}_J(\mathcal{X}\ )$;
  - binary – labelled with an operator $\wedge$, $\mathcal{P}_J(\ \mathcal{U}\ )$, or $\mathcal{P}_J(\ \mathcal{U}^{\leq n})$.

Example: $\neg a \wedge \mathcal{P}_{\leq 0.2}(\neg b\ \ \mathcal{U}\ \ \mathcal{P}_{\geq 0.9}(\Diamond\ c))$



Compute $Sat(\Psi) = \{s \in S \mid s \models \Psi\}$ for each node Ψ of the tree in a bottom-up fashion. Then $s \models \Phi$ iff $s \in Sat(\Phi)$.

PCTL Model Checking - Algorithm - Outline (2)

“Base” of the algorithm:

We need a procedure to compute $Sat(\Psi)$ for Ψ of the form a or $true$:

PCTL Model Checking - Algorithm - Outline (2)

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Lemma

- $Sat(true) = S$,
- $Sat(a) = \{s \mid a \in L(s)\}$

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“Induction” step of the algorithm:

We need a procedure to compute $Sat(\Psi)$ for Ψ given the sets $Sat(\Psi')$ for all state sub-formulas Ψ' of Ψ:

Lemma

- $Sat(\Phi_1 \wedge \Phi_2) =$
- $Sat(\neg\Phi) =$

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Lemma

- $Sat(\Phi_1 \wedge \Phi_2) = Sat(\Phi_1) \cap Sat(\Phi_2)$
- $Sat(\neg\Phi) = S \setminus Sat(\Phi)$

$Sat(\mathcal{P}_J(\Phi)) = \{s \mid P_s(Paths(\Phi)) \in J\}$ discussed on the next slide.

PCTL Model Checking - Algorithm - Path Operator

Lemma

- Next:

$$P_s(Paths(\mathcal{X}\ \Phi)) =$$

- Bounded Until:

$$P_s(Paths(\Phi_1\ \ \mathcal{U}^{\ \leq n}\ \Phi_2)) =$$

- Unbounded Until:

$$P_s(Paths(\Phi_1\ \ \mathcal{U}\ \ \Phi_2)) =$$

PCTL Model Checking - Algorithm - Path Operator

Lemma

- Next:

$$P_s(Paths(\mathcal{X}\ \Phi)) = \sum_{s' \in Sat(\Phi)} \mathrm{P}(s, s')$$

- Bounded Until:

$$P_s(Paths(\Phi_1\ \ \mathcal{U}^{\leq n}\ \Phi_2)) = P_s(Sat(\Phi_1)\ \ \mathcal{U}^{\leq n}\ Sat(\Phi_2))$$

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PCTL Model Checking - Algorithm - Path Operator

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As before:

can be reduced to transient analysis and to unbounded reachability.

PCTL Model Checking - Algorithm - Complexity

Precise algorithm

Computation for every node in the parse tree and for every state:

- All node types except for path operator – trivial.
- Next: Trivial.
- Until: Solving equation systems can be done by polynomially many elementary arithmetic operations.
- Bounded until: Matrix vector multiplications can be done by polynomial many elementary arithmetic operations as well.

Overall complexity:
Polynomial in $|\mathcal{D}|$, linear in $|\Phi|$ and the maximum step bound n.

In practice

The until and bounded until probabilities computed approximatively:

- rounding off probabilities in matrix-vector multiplication,
- using approximative iterative methods (error guarantees?!).

pLTL Model Checking Algorithm

LTL Model Checking - Overview

Definition: LTL Model Checking

Let $\mathcal{D} = (S, \mathrm{P}, \pi_0, L)$ be a DTMC, Ψ a LTL formula, $s \in S$, and $p \in [0, 1]$. The model checking problem is to decide whether $s \models P_s^{\mathcal{D}}(Paths(\Psi)) \geq p$.

Theorem

The LTL model checking can be decided in time $\mathcal{O}(|\mathcal{D}| \cdot 2^{|\Psi|})$.

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1. Construct from Ψ a deterministic Rabin automaton A recognizing words satisfying Ψ, i.e. $Paths(\Psi) := \{L(\omega) \in (2^{Ap})^{\infty} \mid \omega \models \Psi\}$

LTL Model Checking - Overview

Definition: LTL Model Checking

Let $\mathcal{D} = (S, \mathrm{P}, \pi_0, L)$ be a DTMC, Ψ a LTL formula, $s \in S$, and $p \in [0,1]$. The model checking problem is to decide whether $s \models P_s^{\mathcal{D}}(Paths(\Psi)) \geq p$.

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2. Construct a product DTMC $\mathcal{D} \times A$ that “embeds” the deterministic execution of A into the Markov chain.
3. Compute in $\mathcal{D} \times A$ the probability of paths where A satisfies the acceptance condition.

LTL Model Checking - ω-Automata (1.)

Deterministic Rabin automaton (DRA): $(Q, \Sigma, \delta, q_0, Acc)$

- a DFA with a different acceptance condition,
- $Acc = \{(E_i, F_i) \mid 1 \leq i \leq k\}$
- each accepting infinite path must visit for some i
  - all states of E_i at most finitely often and
  - some state of F_i infinitely often.

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Lemma (Vardi&Wolper'86, Safra'88)

For any LTL formula Ψ there is a DRA A recognizing $Paths(\Psi)$ with $|A| \in 2^{2^{\mathcal{O}(|\Psi|)}}$.

LTL Model Checking - Product DTMC (2.)

For a labelled DTMC $\mathcal{D} = (S, \mathrm{P}, \pi_0, L)$ and a DRA $A = (Q, 2^{Ap}, \delta, q_0, \{(E_i, F_i) \mid 1 \leq i \leq k\})$ we define

1. a DTMC $\mathcal{D} \times A = (S \times Q, \mathrm{P}', \pi_0')$:
   - $\mathrm{P}'((s, q), (s', q')) = \mathrm{P}(s, s')$ if $\delta(q, L(s')) = q'$ and 0, otherwise;
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2. $\{(E_i', F_i') \mid 1 \leq i \leq k\}$ where for each i:
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Lemma

The construction preserves probability of accepting as

$$P_s^{\mathcal{D}}(\mathrm{Lang}(A)) = P_{(s,q_s)}^{\mathcal{D}\times A}(\{\omega \mid \exists i : \mathrm{inf}(\omega) \cap E_i' = \emptyset, \mathrm{inf}(\omega) \cap F_i' \neq \emptyset\})$$

where $\mathrm{inf}(\omega)$ *is the set of states visited in* ω *infinitely often.*

Proof sketch.

We have a one-to-one correspondence between executions of $\mathcal{D}$ and $\mathcal{D} \times A$ (as A is deterministic), mapping $\mathrm{Lang}(A)$ to $\{\cdots\}$, and preserving probabilities. $\square$

LTL Model Checking - Computing Acceptance Pr. (3.)

How to check the probability of accepting in $\mathcal{D} \times A$?

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Identify the BSCCs $(C_j)_j$ of $\mathcal{D} \times A$ that for some $1 \leq i \leq k$,

1. contain no state from E_i' and
2. contain some state from F_i'.

Lemma

$P^{\mathcal{D}\times A}_{(s,q_s)}(\{\omega \mid \exists i : \mathrm{inf}(\omega) \cap E_i' = \emptyset, \mathrm{inf}(\omega) \cap F_i' \neq \emptyset\}) = P^{\mathcal{D}\times A}_{(s,q_s)}(\Diamond \bigcup_j C_j)$.

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Proof sketch.

- Note that some BSCC of each finite DTMC is reached with probability 1 *(short paths with prob. bounded from below)*,
- Rabin acceptance condition does not depend on any finite prefix of the infinite word,
- every state of a finite irreducible DTMC is visited infinitely often with probability 1 regardless of the choice of initial state. $\square$

LTL Model Checking – Computing Acceptance Pr. (3.)

How to check the probability of accepting in $\mathcal{D} \times A$?

Identify the BSCCs $(C_j)_j$ of $\mathcal{D} \times A$ that for some $1 \leq i \leq k$,

1. contain no state from E'_i and
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Lemma

$P^{\mathcal{D}\times A}_{(s,q_s)}(\{\omega \mid \exists i : \inf(\omega) \cap E'_i = \emptyset, \inf(\omega) \cap F'_i \neq \emptyset\}) = P^{\mathcal{D}\times A}_{(s,q_s)}(\Diamond \bigcup_j C_j)$.

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Corollary

$P^{\mathcal{D}}_{s}(\mathrm{Lang}(A)) = P^{\mathcal{D}\times A}_{(s,q_s)}(\Diamond \bigcup_j C_j)$.

LTL Model Checking - Algorithm - Complexity

Doubly exponential in Ψ and polynomial in $\mathcal{D}$
(for the algorithm presented here):

1. $|A|$ and hence also $|\mathcal{D} \times A|$ is of size $2^{2^{\mathcal{O}(|\Psi|)}}$
2. BSCC computation: Tarjan algorithm - linear in $|\mathcal{D} \times A|$
   (number of states + transitions)
3. Unbounded reachability: system of linear equations ($\leq |\mathcal{D} \times A|$):
   - exact solution: $\approx$ cubic in the size of the system
   - approximative solution: efficient in practice