Recent Advances in Model Checking

Practical Course

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Motivation

Example I: Simulation of a die by coins

Knuth & Yao die

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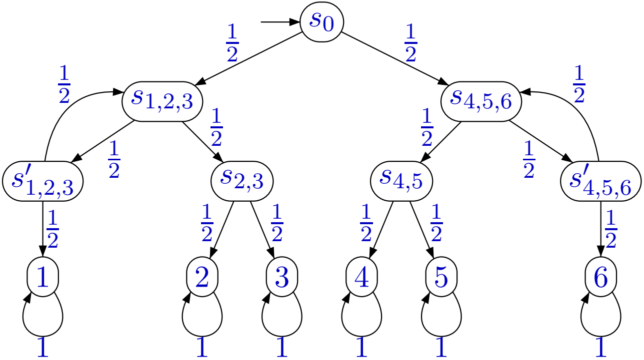
Knuth & Yao die



Question:

- What is the probability of obtaining 2?

Example II: Zero Configuration Networking (Zeroconf)

- Previously: Manual assignment of IP addresses
- Zeroconf: Dynamic configuration of local IPv4 addresses
- Advantage: Simple devices able to communicate automatically

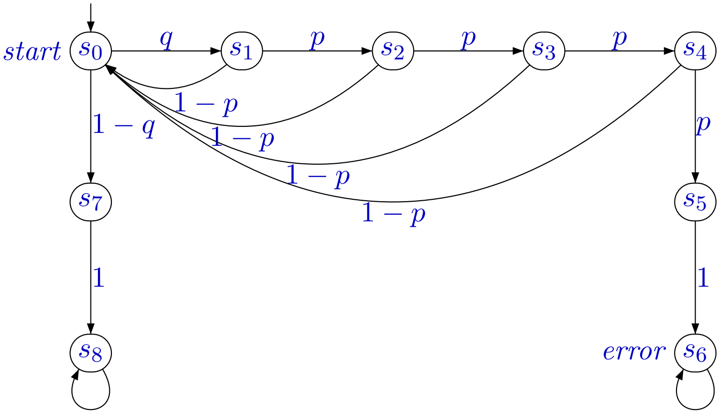
Automatic Private IP Addressing (APIPA) – RFC 3927

- Used when DHCP is configured but unavailable
- Pick randomly an address from 169.254.1.0 – 169.254.254.255
- Find out whether anybody else uses this address (by sending several ARP requests)

Model:

- Randomly pick an address among the K (65024) addresses.
- With m hosts in the network, collision probability is $q = \frac{m}{K}$.
- Send 4 ARP requests.
- In case of collision, the probability of no answer to the ARP request is p (due to the lossy channel)

Example II: Zero Configuration Networking (Zeroconf)



For 100 hosts and $p = 0.001$, the probability of error is $\approx 1.55 \cdot 10^{-15}$.

Application I – Non-deterministic Systems

- Verification of non-deterministic systems
- Controller synthesis for under-specified systems

Given a model S of a system and formula ϕ, the model checking problem is to decide whether $K \models \phi$ (for all/some resolutions of choices).

Application I – Non-deterministic Systems

- Verification of non-deterministic systems
- Controller synthesis for under-specified systems

Given a model S of a system and formula ϕ, the model checking problem is to decide whether $K \models \phi$ (for all/some resolutions of choices).

Solution: Combine K and ϕ into a "product game graph" $K \times \phi$ with a "winning condition" such that

$$K \models \phi$$

iff

from a designated vertex of $K \times \phi$ player 0 has "winning strategy".

Aplication II – Synthesis

Alonzo Church, 1957

“Given a requirement which a circuit is to satisfy, we may suppose the requirement expressed in some suitable logistic system which is an extension of restricted recursive arithmetic. The synthesis problem is then to find recursion equivalences representing a circuit that satisfies the given requirement (or alternatively, to determine that there is no such circuit).”

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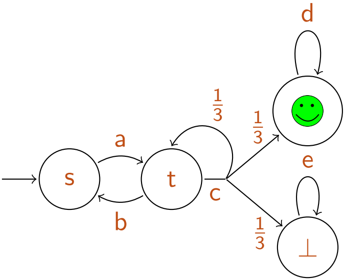
Given a requirement on a bit stream transformation

$$\xrightarrow[\ldots 1011]{\text{input}} \boxed{\qquad\qquad} \xrightarrow[\ldots 0100]{\text{output}}$$

fill the box by a machine with output, satisfying the requirement (or state that the requirement is not satisfiable).

Discrete-time Markov Decision Processes MDP

MDP



Markov chains – purely probabilistic

Possible successor states are chosen based on probabilities but not on decisions.

We want decisions

to model both

- controllable setting (game theory, operations theory, control theory);
- uncontrollable setting (interleaving in concurrent systems, abstractions of models, open systems)

MDP: Definition

Definition:
A (labelled) Markov Decision Process (MDP) is a tuple

$$\mathcal{M} = (S, Act, \mathrm{P}, \pi_0)$$

where

- S is a countable set of states,
- Act is a finite set of actions,
- $\mathrm{P} : S \times Act \times S \to [0, 1]$ is the transition probability function, such that for each state s and action α,
  - $\sum_{s' \in S} \mathrm{P}(s, \alpha, s') = 1$, then we say that α is enabled in s; or
  - $\mathrm{P}(s, \alpha, s') = 0$ for all s', then we say that α is not enabled in s.
- π_0 is the initial distribution.

The set of actions enabled in s is denoted by $Act(s)$. We assume that for each s, we have $Act(s) \neq \emptyset$.

MDP – Schedulers

Problem:

How is the non-determinism resolved?
(Possibly allowing also for memory and randomness)

Definition (Scheduler):

A scheduler (also called strategy or policy) on an MDP $\mathcal{M} = (S, Act, \mathsf{P}, \pi_0)$ is a function Θ assigning to each state $s \in S$ an action α that is enabled in s.

Definition (Induced DTMC):

Let $\mathcal{M} = (S, Act, \mathsf{P}, \pi_0)$ be a MDP and scheduler Θ on $\mathcal{M}$. The induced DTMC is given by

$$\mathcal{M}^{\Theta} = (S, \mathsf{P}^{\Theta}, \pi_0),$$

where

$$\mathsf{P}^{\Theta}(s, s') = \mathsf{P}(s, \Theta(s), s')$$

MDP – General Schedulers

Definition (Scheduler):

A scheduler (also called strategy or policy) on an MDP $\mathcal{M} = (S, Act, \mathrm{P}, \pi_0)$ is a function Θ assigning to each history $s_0 \cdots s_n \in S^+$ a probability distribution over Act such that α is enabled in s_n whenever $\Theta(s_0 \cdots s_n)(\alpha) > 0$.

Definition (Induced DTMC):

Let $\mathcal{M} = (S, Act, \mathrm{P}, \pi_0)$ be a MDP and scheduler Θ on $\mathcal{M}$. The induced DTMC is given by

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where for any $h = s_0 s_1 \ldots s_n$, we define

$$\mathrm{P}^\Theta(h, hs_{n+1}) = \sum_{\alpha \in Act} \Theta(h)(\alpha) \cdot \mathrm{P}(s_n, \alpha, s_{n+1})$$

MDP – Reachability

MDP - Reachability

Min

When playing “Mensch Ärgere dich nicht” against a fixed opponent strategy, what is the minimal probability of having all pieces kicked out into the outside area?

Max

What is the maximal probability of winning the game?

MDP - Reachability

Min

- Best case for reaching undesirable states when controlled
- Worst case for reaching desirable states when not controlled

The minimum probability to reach a set of states B from a state s (within n steps) is

$$\inf_{\Theta} P_s^{\Theta}(\Diamond B), \qquad \inf_{\Theta} P_s^{\Theta}(\Diamond^{\leq n} B)$$

Max

- Best case for reaching desirable states when controlled
- Worst case for reaching undesirable states when not controlled

The maximum probability to reach a set of states B from a state s (within n steps) is

$$\sup_{\Theta} P_s^{\Theta}(\Diamond B), \qquad \sup_{\Theta} P_s^{\Theta}(\Diamond^{\leq n} B)$$

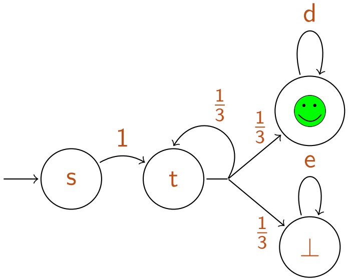
Focus on maximum; minimum is similar

MDP - Reachability

Recall for Markov chains

Let (S, P, π_0) be a finite DTMC and $B \subseteq S$. The vector x with $\mathrm{x}(s) = P_s(\Diamond B)$ is the unique solution of the equation system

$$\mathrm{x}(s) = \begin{cases} 1 & \text{if } s \in B, \\ 0 & \text{if } s \in S_0 = \{s \mid P_s(\Diamond B) = 0\}, \\ \sum_{s' \in S} \mathrm{P}(s, s') \cdot \mathrm{x}(s') & \text{otherwise.} \end{cases}$$



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Theorem (Maximum Reachability Probability):

Let $(S, Act, \mathrm{P}, \pi_0)$ be a finite MDP and $B \subseteq S$. The vector x with $\mathrm{x}(s) = \sup_\Theta P_s^\Theta(\Diamond B)$ is the least solution of the equation system

$$\mathrm{x}(s) = \begin{cases} 1 & \text{if } s \in B, \\ 0 & \text{if } s \in S_0^{max} = \{s \mid \sup_\Theta P_s^\Theta(\Diamond B) = 0\}, \\ \max_{\alpha \in Act(s)} \sum_{s' \in S} \mathrm{P}(s, \alpha, s') \cdot \mathrm{x}(s') & \text{otherwise.} \end{cases}$$

MDP - Reachability - Linear Programming

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Linear Program:

Let $(S, Act, \mathrm{P}, \pi_0)$ be a finite MDP and $B \subseteq S$. The vector x with $\mathrm{x}(s) = \max_{\Theta} P_s^{\Theta}(\Diamond B)$ is the unique solution of the linear program

satisfying
$$\begin{aligned}
\mathrm{x}(s) &= 1 && \forall s \in B,\\
\mathrm{x}(s) &= 0 && \forall s \in S_0^{\max},\\
\mathrm{x}(s) &\geq \sum_{u \in S} \mathrm{P}(s, \alpha, u) \cdot \mathrm{x}(u) && \forall s \in S \setminus (B \cup S_0^{\max}), \forall \alpha \in Act.
\end{aligned}$$

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$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{s \in S} \mathrm{x}(s) & \\
\text{satisfying} & \mathrm{x}(s) = 1 & \forall s \in B, \\
 & \mathrm{x}(s) = 0 & \forall s \in S_0^{\max}, \\
 & \mathrm{x}(s) \geq \displaystyle\sum_{u \in S} \mathrm{P}(s, \alpha, u) \cdot \mathrm{x}(u) & \forall s \in S \setminus (B \cup S_0^{\max}), \forall \alpha \in Act.
\end{array}
$$

MDP - Reachability - Value Iteration

Value Iteration Algorithm:

Let $\mathcal{M}$ be a finite MDP with state space S, and $B \subseteq S$.

- Initialize $\mathrm{x}_0(s)$ to 1 if $s \in B$ and to 0, otherwise.
- Iterate

$$\mathrm{x}_{n+1}(s) = \begin{cases} 1 & \text{if } s \in B, \\ 0 & \text{if } s \in S_0^{\max}, \\ \max\limits_{\alpha \in Act(s)} \sum\limits_{s' \in S} \mathrm{P}(s, \alpha, s') \cdot \mathrm{x}_n(s') & \text{otherwise} \end{cases}$$

until convergence.
I.e., until $\max_{s \in S} |\mathrm{x}_{n+1}(s) - \mathrm{x}_n(s)| < \epsilon$ for a small $\epsilon > 0$?

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Theorem

- $\mathrm{x}_n(s) = \sup\limits_{\Theta} P_s^{\Theta}(\Diamond^{\leq n} B)$.
- $\mathrm{x}_{n+1} \geq \mathrm{x}_n$.
- $\lim\limits_{n \to \infty} \mathrm{x}_n(s) = \sup\limits_{\Theta} P_s^{\Theta}(\Diamond B)$.

MDP - Reachability - Computing $S_0^{\max}$

We rather compute the set

$$S_{>0}^{\max} = \{s \mid \sup_{\Theta} P_s^{\Theta}(\Diamond B) > 0\}$$

and return

$$S_0^{\max} = S \setminus S_{>0}^{\max}$$

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$S_{>0}^{\max}$:

Initialize the set to B and in every iteration add states that reach the set in one step with positive probability for some enabled action. Repeat until fix-point is reached.

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$S_{>0}^{\max}$:
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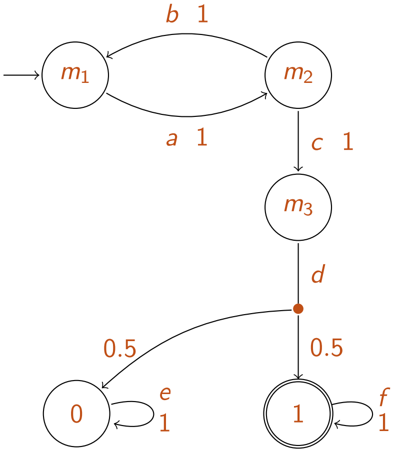
(Similarly for $S_{>0}^{\min}$: replace “some” by ”every”)

General MDP with **end components**



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