Lecture 9

Normal forms for first-order logic

Equivalences, prenex form, Skolem form

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(with small changes by Javier Esparza)

Recap

Syntax of first-order formulas:

- Signature σ (constant, function and predicate symbols)
- σ-terms
- Formulas (as in propositional logic, predicate symbols atomic formulas, additional $\forall x$ and $\exists x$)

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- σ-structure $\mathcal{A}$ with universe $U_{\mathcal{A}}$ and interpretations of constants, functions, predicates, and variables
- $\mathcal{A} \models F$ defined by structural induction on F

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Syntax of first-order formulas:

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Semantics of first-order formulas:

- σ-structure $\mathcal{A}$ with universe $U_{\mathcal{A}}$ and interpretations of constants, functions, predicates, and variables
- $\mathcal{A} \models F$ defined by structural induction on F

Relevance lemma: “If $\mathcal{A}$ and $\mathcal{A}'$ only differ on variables other than free variables in F, then $\mathcal{A} \models F$ if and only if $\mathcal{A}' \models F$.”

Normal forms

$$\exists x \left(\neg(\exists x\, P(x, y) \vee \forall y\, \neg Q(y)) \wedge Q(x)\right)$$

vs

$$\forall x\, \exists z\, \exists w\, ((\neg P(x, y) \wedge Q(z)) \wedge Q(w))$$

vs
(with convention on parenthesis)

$$\forall x\, \exists z\, \exists w\, (\neg P(x, y) \wedge Q(z) \wedge Q(w))$$

Normal forms

$$\exists x\,\big(\neg(\exists x\,P(x,y)\lor\forall y\,\neg Q(y))\land Q(x)\big)$$

vs

$$\forall x\,\exists z\,\exists w\,((\neg P(x,y)\land Q(z))\land Q(w))$$

vs

(with convention on parenthesis)

$$\forall x\,\exists z\,\exists w\,(\neg P(x,y)\land Q(z)\land Q(w))$$

This lecture:

- Establish elementary equivalences
- Rectified form: “different variables have different names”
- Prenex form: all quantifiers first
- Skolem form: prenex form with no existential quantifiers

Equivalences

Definition

Two first-order logic formulas F and G over the signature σ are **logically equivalent** (written $F \equiv G$) if $\mathcal{A} \models F$ iff $\mathcal{A} \models G$ for all σ-assignments $\mathcal{A}$.

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Two first-order logic formulas F and G over the signature σ are **logically equivalent** (written $F \equiv G$) if $\mathcal{A} \models F$ iff $\mathcal{A} \models G$ for all σ-assignments $\mathcal{A}$.

Proposition

Let F and G be arbitrary formulas. Then

(A) $\neg\forall x F \equiv \exists x \neg F$ and $\neg\exists x F \equiv \forall x \neg F$

(B) If x does not occur free in G then:

$$(\forall x F \wedge G) \equiv \forall x(F \wedge G) \qquad (\forall x F \vee G) \equiv \forall x(F \vee G)$$
$$(\exists x F \wedge G) \equiv \exists x(F \wedge G) \qquad (\exists x F \vee G) \equiv \exists x(F \vee G)$$

(C) $(\forall x F \wedge \forall x G) \equiv \forall x(F \wedge G)$ and $(\exists x F \vee \exists x G) \equiv \exists x(F \vee G)$

(D) $\forall x \forall y F \equiv \forall y \forall x F$ and $\exists x \exists y F \equiv \exists y \exists x F$

Proof.

We only sketch the proof of the first equivalence in (B).

$$\mathcal{A} \models (\forall x F \wedge G)$$

Equivalences

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\begin{aligned}
& \mathcal{A} \models (\forall x F \wedge G) \\
\text{iff} \quad & \mathcal{A} \models \forall x F \text{ and } \mathcal{A} \models G
\end{aligned}
$$

Equivalences

Proof.

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$$\mathcal{A} \models (\forall x F \wedge G)$$

iff $\mathcal{A} \models \forall x F$ and $\mathcal{A} \models G$

iff for all $a \in U_{\mathcal{A}}$, $\mathcal{A}_{[x \mapsto a]} \models F$ and $\mathcal{A} \models G$

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\text{iff} & \text{for all } a \in U_{\mathcal{A}},\ \mathcal{A}_{[x \mapsto a]} \models F \wedge G \\
\text{iff} & \mathcal{A} \models \forall x (F \wedge G)
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□

Translation lemma

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Lemma (Translation Lemma)

If t is a term and F is a formula such that no variable in t occurs bound in F, then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

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Proof.

By structural induction on formulas. □

Translation lemma: One case of the proof

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We consider only one case of the proof, namely the one where the formula is of the form $F \equiv \forall y\ G$, where $y \neq x$.

$$\mathcal{A} \models (\forall y G)[t/x]$$

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\text{iff} & \mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models \forall y G &
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Rectified formulas

Definition

A formula is **rectified** if no variable occurs both bound and free, and if all quantifiers in the formula refer to different variables.

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Proposition

Let $Qx\,G$ be a formula where $Q \in \{\forall, \exists\}$, and let y be a variable that does not occur in G. Then $Qx\,G \equiv Qy\,(G[y/x])$.

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Proof.

Proof for $\forall$:

$$
\begin{array}{ll}
 & \mathcal{A} \models \forall y\,(G[y/x]) \\
\text{iff} & \mathcal{A}_{[y\mapsto a]} \models G[y/x] \text{ for all } a \in U_{\mathcal{A}} \\
\text{iff} & \mathcal{A}_{[y\mapsto a][x\mapsto \mathcal{A}_{[y\mapsto a]}(y)]} \models G \text{ for all } a \in U_{\mathcal{A}} \text{ (Translation Lemma)} \\
\text{iff} & \mathcal{A}_{[y\mapsto a][x\mapsto a]} \models G \text{ for all } a \in U_{\mathcal{A}} \\
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\text{iff} & \mathcal{A} \models \forall x\, G\,. \quad \square
\end{array}
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Proposition

Every formula is equivalent to a rectified formula.

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Proof.

Repeatedly apply the previous proposition to replace bound occurrences of variables by fresh variables not occurring in the original formula. □

Prenex form

Definition

A formula is in **prenex form** if it is of the form

$$Q_1 y_1\, Q_2 y_2 \ldots Q_n y_n\, F,$$

where $Q_i \in \{\exists, \forall\}$, $n \geq 0$, and F contains no quantifiers. In this case F is called the **matrix** of the formula.

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Theorem

Every formula is equivalent to a formula in rectified prenex form.

Proof (sketch).

- Rectify formula
- Move all quantifiers up the syntax tree using the equivalences (A) and (B), plus equivalences of propositional logic. □

Prenex form

Example

$$
\begin{aligned}
& \exists x\,\big(\neg(\exists x\, P(x,y) \lor \forall y\, \neg Q(y)) \land Q(x)\big) \\
\equiv\; & \exists x\,\big(\neg(\exists x\, P(x,y) \lor \forall z\, \neg Q(z)) \land Q(x)\big) \\
\equiv\; & \exists w\,\big(\neg(\exists x\, P(x,y) \lor \forall z\, \neg Q(z)) \land Q(w)\big) \\
\equiv\; & \exists w\,\big((\neg\exists x\, P(x,y) \land \neg\forall z\, \neg Q(z)) \land Q(w)\big) \\
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\equiv\; & \exists z\, \exists w\,\big((\forall x\, \neg P(x,y) \land Q(z)) \land Q(w)\big) \\
\equiv\; & \forall x\, \exists z\, \exists w\,\big(\neg P(x,y) \land Q(z) \land Q(w)\big)
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Skolem form

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Proposition

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n \exists z\, G$ be a rectified formula. Given a function symbol f of arity n that does not occur in F, write

$$F' = \forall y_1 \forall y_2 \ldots \forall y_n\, G[f(y_1, \ldots, y_n)/z] \,.$$

Then F and F' are equisatisfiable.

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A formula is in **Skolem form** if it is in rectified prenex form and no existential quantifier occurs in it.

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Then F and F' are equisatisfiable.

"Proof".

Choose assignment $\mathcal{A}'$ for F' such that $\mathcal{A}'$ "emulates" via f the choice made by existential quantifier. □

Skolem form

Theorem

Every formula of first-order logic has an equisatisfiable formula in Skolem form.

Proof.

Put the formula in rectified prenex form. Repeatedly apply the previous proposition to the **outermost** existential quantifier in the block of quantifiers. □

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Example

$$\begin{aligned} & \forall x\, \exists y\, \forall z\, \exists w\, (\neg P(a, w) \vee Q(f(x), y)) \text{ is satisfiable} \\ \text{iff} \quad & \forall x\, \forall z\, \exists w\, (\neg P(a, w) \vee Q(f(x), g(x))) \text{ is satisfiable} \\ \text{iff} \quad & \forall x\, \forall z\, (\neg P(a, h(x, z)) \vee Q(f(x), g(x))) \text{ is satisfiable} \end{aligned}$$

Clause form

Definition

A closed formula is in **clause form** if it is of the form

$$\forall y_1 \forall y_2 \ldots \forall y_n\, F$$

where F contains no quantifiers and is in **CNF**.

A closed formula in clause form can be represented as a set of clauses.

Example: the clause form of $\forall x \forall y\; ((P(x, y) \wedge Q(x)) \wedge P(f(y), a)$ is

$$\{\ \{P(x, y), Q(x)\}\ ,\ \{P(f(y), a)\}\ \}$$

Converting into clause form up to equisatisfiability

Given: a formula F of predicate logic (with possible occurrences of free variables).

1. Rectify F by systematic renaming of bound variables.
   The result is a formula F_1 equivalent to F.
2. Let $y_1, y_2, \ldots, y_n$ be the variables occurring free in F_1.
   Produce the formula $F_2 = \exists y_1 \exists y_2 \ldots \exists y_n F_1$.
   F_2 is equisatisfiable to F_1 and closed.
3. Produce a formula F_3 in prenex form equivalent to F_2.
4. Eliminate the existential quantifiers in F_3 by transforming F_3 into a Skolem formula F_4.
   The formula F_4 is equisatisfiable to F_3.
5. Convert the matrix of F_4 into **CNF** (and write the resulting formula F_5 as set of clauses).

Exercise

Which formulas are rectified, in prenex, Skolem, or clause form?

	R	P	S	C
$\forall x(\mathit{Tet}(x) \lor \mathit{Cube}(x) \lor \mathit{Dodec}(x))$				
$\exists x \exists y(\mathit{Cube}(y) \lor \mathit{BackOf}(x, y))$				
$\forall x(\neg \mathit{FrontOf}(x, x) \land \neg \mathit{BackOf}(x, x))$				
$\neg \exists x \mathit{Cube}(x) \leftrightarrow \forall x \neg \mathit{Cube}(x)$				
$\forall x(\mathit{Cube}(x) \to \mathit{Small}(x)) \to \forall y(\neg \mathit{Cube}(y) \to \neg \mathit{Small}(y))$				
$(\mathit{Cube}(a) \land \forall x \mathit{Small}(x)) \to \mathit{Small}(a)$				
$\exists x(\mathit{Larger}(a, x) \land \mathit{Larger}(x, b)) \to \mathit{Larger}(a, b)$				