Lecture 8

First-order logic

Syntax and semantics

Dr Christoph Haase
University of Oxford
(with small changes by Javier Esparza)

Limitations of propositional logic

- Can only reason about true or false
- Atomic formulas have no internal structure
- Impossible to express “real” mathematical statements

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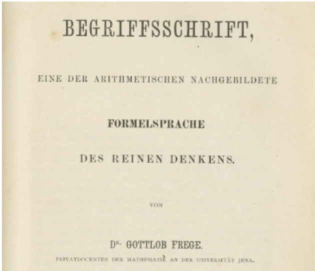
Example

Every natural number x is either odd or even.

Frege's Begriffsschrift


BEGRIFFSSCHRIFT,

EINE DER ARITHMETISCHEN NACHGEBILDETE

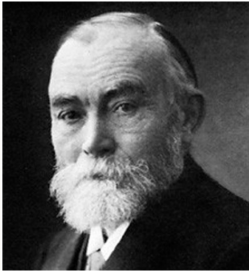
FORMELSPRACHE

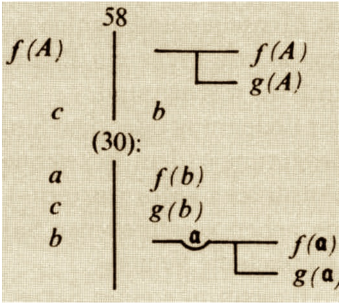
DES REINEN DENKENS.

VON

DR. GOTTLOB FREGE,

PRIVATDOCENTEN DER MATHEMATIK AN DER UNIVERSITÄT JENA.





Signatures

Definition

A **signature** σ is a tuple consisting of

- a set of **constant symbols** (denoted c, d)
- a set of **function symbols** (denoted f, g), and
- a set of **predicate symbols** (denoted P, Q, R).

Each function and predicate symbol has an **arity** $k \geq 1$.

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Example

The signature of number theory is $\sigma = \langle 0, 1, +, \cdot, = \rangle$, where 0 and 1 are constant symbols, $+$ and $\cdot$ are function symbols of arity two, and $=$ is a predicate symbol of arity two.

Terms

Definition

Let $\mathcal{X}$ be a countably infinite set of **variables** (denoted x, y, z. The **terms** over a signature σ are defined by structural induction:

- Each variable $x \in \mathcal{X}$ is a term.
- Each constant symbol c is a term.
- If $t_1, \ldots, t_k$ are terms and f is a k-ary function symbol then $f(t_1, \ldots, t_k)$ is a term.

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Example

Given the signature of number theory σ, we have that $\cdot(+(1, 1), x)$ is a term. We often use **infix** notation and write $(1 + 1) \cdot x$ instead.

Formulas

Definition

The set of **formulas** over a signature σ is defined inductively:

- Given terms $t_1, \ldots, t_k$ and a k-ary predicate symbol P then $P(t_1, \ldots, t_k)$ is a formula (**atomic formulas**).
- For each formula F, $\neg F$ is a formula.
- For each pair of formulas F, G, $(F \vee G)$ and $(F \wedge G)$ are both formulas.
- If F is a formula and x is a variable then $\exists x\, F$ and $\forall x\, F$ are both formulas.

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$\exists$ and $\forall$ are the **existential** and **universal (first-order) quantifiers**.

Example

A formula over the signature of number theory::

$$\forall x\, \exists y (=(x, ((1+1) \cdot y))) \vee (=(x, (1 + (((1+1) \cdot y))))).$$

Again, use infix notation also for predicate symbols:

$$\forall x\, \exists y ((x = (1+1) \cdot y) \vee (x = 1 + (1+1) \cdot y)).$$

Quantifier depth and bounded variables

Inductive structure of formulas enables structural induction:

Definition

quantifier depth is defined as follows:

$$\begin{aligned}
\mathrm{qd}(P(t_1, \ldots, t_k)) &:= 0 \\
\mathrm{qd}(\neg F) &:= \mathrm{qd}(F) \\
\mathrm{qd}(F \wedge G) = \mathrm{qd}(F \vee G) &:= \max(\mathrm{qd}(F), \mathrm{qd}(G)) \\
\mathrm{qd}(\exists x\, F) = \mathrm{qd}(\forall x\, F) &:= \mathrm{qd}(F) + 1.
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Definition

In formula $\exists x\, G$, we say G is in the **scope** of the quantifier $\exists x$, likewise for $\forall x\, G$. A variable x is **bound** in F if x occurs in scope of $\exists x$ or $\forall x$. If x is not bound then x is **free**. Formula with no free variables is called **closed** or **sentence**.

Exercise

NF: non-formula F: formula, but not closed C: closed

	NF	F	C
$\forall x\ P(c)$			
$\forall x \exists y\ (Q(x, y) \vee R(x, y))$			
$\forall x\ Q(x, x) \rightarrow \exists x\ Q(x, y)$			
$\forall x\ P(x) \vee \forall x\ Q(x, x)$			
$\forall x\ (P(y) \wedge \forall y\ P(x))$			
$P(x) \rightarrow \exists x\ Q(x, P(x))$			
$\forall f\ \exists x\ P(f(x))$			

Exercise

NF: non-formula F: formula, but not closed C: closed

	NF	F	C
$\forall x\ (\neg\forall y\ Q(x,y) \wedge R(x,y))$			
$\exists z\ (Q(z,x) \vee R(y,z)) \rightarrow \exists y\ (R(x,y) \wedge Q(x,z))$			
$\exists x\ (\neg P(x) \vee P(f(c)))$			
$P(x) \rightarrow \exists x\ P(x)$			
$\exists x \forall y\ ((P(y) \rightarrow Q(x,y)) \vee \neg P(x))$			
$\exists x \forall x\ Q(x,x)$			

Semantics of first-order logic

Definition

Given a signature σ, a σ**-structure** (or **assignment**) $\mathcal{A}$ consists of:

- a non-empty set $U_\mathcal{A}$ called the **universe** of the structure;
- for each k-ary predicate symbol P in σ, a k-ary relation

$$P_\mathcal{A} \subseteq \underbrace{U_\mathcal{A} \times \cdots \times U_\mathcal{A}}_{k};$$

- for each k-ary function symbol f in σ, a k-ary function,

$$f_\mathcal{A} \colon \underbrace{U_\mathcal{A} \times \cdots \times U_\mathcal{A}}_{k} \to U_\mathcal{A};$$

- for each constant symbol c, an element $c_\mathcal{A}$ of $U_\mathcal{A}$;
- for each variable x an element $x_\mathcal{A}$ of $U_\mathcal{A}$.

Example

Let σ be the signature of number theory. The natural σ-structure $\mathcal{A}$ is:

- $U_{\mathcal{A}} := \mathbb{N} = \{0, 1, \ldots\}$
- $0_{\mathcal{A}} := 0$, $1_{\mathcal{A}} := 1$
- $+_{\mathcal{A}} := (m, n) \mapsto m + n$
- $\cdot_{\mathcal{A}} := (m, n) \mapsto m \cdot n$
- $=_{\mathcal{A}} := \{(n, n) : n \in \mathbb{N}\}$

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- $=_{\mathcal{A}} := \{(n, n) : n \in \mathbb{N}\}$

BUT the following $\mathcal{B}$ is also a σ-structure:

- $U_{\mathcal{B}} := \{A, B, 5\}$
- $0_{\mathcal{B}} := A$, $1_{\mathcal{A}} := 5$
- $+_{\mathcal{B}} := (m, n) \mapsto 5$
- $\cdot_{\mathcal{B}} := (m, n) \mapsto A$
- $=_{\mathcal{B}} := \{(A, B), (B, B)\}$

Semantics of first-order logic

Definition

Value $\mathcal{A}(t) \in U_{\mathcal{A}}$ of term t inductively defined as follows:

- For a constant symbol c, $\mathcal{A}(c) := c_{\mathcal{A}}$.
- For a variable x, $\mathcal{A}(x) := x_{\mathcal{A}}$.
- For a term $f(t_1, \ldots, t_k)$, where f is k-ary function symbol and $t_1, \ldots, t_k$ are terms,

$$\mathcal{A}(f(t_1, \ldots, t_k)) := f_{\mathcal{A}}(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_k)).$$

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Definition

Define the **satisfaction relation** $\mathcal{A} \models F$ ($\mathcal{A}$ **satisfies** F, or $\mathcal{A}$ is a **model** of F) by structural induction:

- $\mathcal{A} \models P(t_1, \ldots, t_k)$ if and only if $(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_k)) \in P_{\mathcal{A}}$.
- $\mathcal{A} \models (F \wedge G)$ if and only if $\mathcal{A} \models F$ and $\mathcal{A} \models G$.
- $\mathcal{A} \models (F \vee G)$ if and only if $\mathcal{A} \models F$ or $\mathcal{A} \models G$.
- $\mathcal{A} \models \neg F$ if and only if $\mathcal{A} \not\models F$.
- $\mathcal{A} \models \exists x\, F$ if and only if there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \models F$.
- $\mathcal{A} \models \forall x\, F$ if and only if $\mathcal{A}_{[x \mapsto a]} \models F$ for all $a \in U_{\mathcal{A}}$.

Semantics of first-order logic

Example

Let $\mathcal{A}$ be the natural σ-structure of number theory, then

$$\mathcal{A} \models \forall x \, \exists y ((x = (1 + 1) \cdot y) \vee (x = 1 + (1 + 1) \cdot y)).$$

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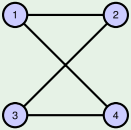
Example



- Undirected graph as σ-structure with one binary relation symbol E interpreted as the edge relation.
- Above graph represented by structure $\mathcal{A}$ with universe $U_{\mathcal{A}} = \{1, 2, 3, 4\}$ and irreflexive symmetric binary relation

$$E_{\mathcal{A}} = \{(1,2), (2,3), (3,4), (4,1), (2,1), (3,2), (4,3), (1,4)\}\,.$$

- Edge relation is irreflexive and symmetric:

$$\forall x\, \neg E(x,x) \wedge \forall x\, \forall y\, (E(x,y) \rightarrow E(y,x))\,.$$

- Every pair of nodes are connected by a path of length 3:

$$\forall x\, \forall y\, \exists z_1\, \exists z_2\, (E(x,z_1) \wedge E(z_1,z_2) \wedge E(z_2,y)).$$

The relevance lemma

Lemma

Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are σ-assignments with the same universe and identical interpretations of the predicate, function, and constant symbols in σ. If $\mathcal{A}$ and $\mathcal{A}'$ give the same interpretation to each variable occurring free in some σ-formula F then $\mathcal{A} \models F$ if and only if $\mathcal{A}' \models F$.

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Proof.

By structural induction on terms and formulas. □

Validity, satisfiability, consequence, equivalence

Let F be a formula over a signature σ.

1. F ist **valid** if every σ-structure is a model of F, denoted $\models F$.
2. F is **satisfiable** if it has at least one model.

Let $F_1, \ldots, F_k, G$ be formulas over the same signature.

G is a **consequence of** or **entailed by** $F_1, \ldots, F_k$, denoted $F_1, \ldots, F_k \models G$, if every model of $\{F_1, \ldots, F_k\}$ is also model of G.

Two formulas F and G over the same signature are **equivalent**, denoted $F \equiv G$, if they have the same models.

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x\, P(a)$			
$\exists x\, (\neg P(x) \vee P(a))$			
$P(a) \to \exists x\, P(x)$			
$P(x) \to \exists x\, P(x)$			
$\forall x\, P(x) \to \exists x\, P(x)$			
$\forall x\, P(x) \wedge \neg \forall y\, P(y)$			

Exercise

V: valid S: satisfiable, but not valid U: unsatisfiable

	V	S	U
$\forall x \ (P(x,x) \to \exists x \forall y \ P(x,y))$			
$\forall x \forall y \ (x = y \to f(x) = f(y))$			
$\forall x \forall y \ (f(x) = f(y) \to x = y)$			
$\exists x \exists y \exists z \ (f(x) = y \land f(x) = z \land y \neq z)$			

Exercise

1. $\forall x\ P(x) \vee \forall x\ Q(x,x)$
2. $\forall x\ (P(x) \vee Q(x,x))$
3. $\forall x\ (\forall z P(z) \vee \forall y\ Q(x,y))$

	Y	N
$1 \models 2$		
$2 \models 3$		
$3 \models 1$		

Exercise

1. $\exists y \forall x \; P(x, y)$
2. $\forall x \exists y \; P(x, y)$

	Y	N
$1 \models 2$		
$2 \models 1$		

Exercise

	Y	N
$\forall x \forall y\ F \equiv \forall y \forall x\ F$		
$\forall x \exists y\ F \equiv \exists x \forall y\ F$		
$\exists x \exists y\ F \equiv \exists y \exists x\ F$		
$\forall x\ F \vee \forall x\ G \equiv \forall x\ (F \vee G)$		
$\forall x\ F \wedge \forall x\ G \equiv \forall x\ (F \wedge G)$		
$\exists x\ F \vee \exists x\ G \equiv \exists x\ (F \vee G)$		
$\exists x\ F \wedge \exists x\ G \equiv \exists x\ (F \wedge G)$		

Predicate logic with equality

Predicate logic
$+$
distinguished predicate symbol "$=$" of arity 2.

Semantics: a structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol $=$ to the identity relation:

$$\mathcal{A}(=) \; = \; \{(d, d) \mid d \in U_{\mathcal{A}}\} \, .$$

Formalizing statements

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It means to give

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such that the statement holds iff **every** σ **structure that extends** $\mathcal{A}$ is a model of F.

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Intuitively, the symbols interpreted in $\mathcal{A}$ are those that the formalizer assumes are known by whoever is going to read the formula. *F* may contain other symbols, but then *F* must define what they mean (see next slides).

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We give different formalizations of the statement

There are infinitely many prime numbers

Formalization I

If the meanings of “prime” and “‘greater-than” are known, then we can take a signature with a unary predicate symbol *Pr* and a binary predicate symbol $>$:

Formula F_1: $\forall x \exists y \, (Pr(y) \wedge y > x)$

Structure $\mathcal{A}_1$:

$$\begin{array}{rcl} U_{\mathcal{A}_1} & = & \mathbb{N} \\ Pr_{\mathcal{A}_1} & = & \{n \in \mathbb{N} \mid n \text{ is prime}\} \\ >_{\mathcal{A}_1} & = & \{(n, m) \in \mathbb{N} \mid n > m\} \end{array}$$

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What if the meaning of “prime” is not known?

Formalization II

If the meaning of “divides” is known, then we can take a signature with a constant *one* and two binary predicate symbols $>$, *Dv* (we use predicate logic with equality), and **define** “prime”:

Formula F_2:

$$\forall x\ Pr(x) \leftrightarrow (\forall y\ Dv(y,x) \rightarrow (y = x \vee y = one))$$
$$\rightarrow\ \ \forall x \exists y\ Pr(y) \wedge y > x$$

Structure $\mathcal{A}_2$:

$$U_{\mathcal{A}_2} = \mathbb{N}$$
$$Dv_{\mathcal{A}_2} = \{(n,n) \in \mathbb{N} \mid n \text{ divides } m\}$$
$$>_{\mathcal{A}_2} = \{(n,m) \in \mathbb{N} \mid n > m\}$$

We are now stating “ if we define prime numbers as . . . then there are infinitely many prime numbers”.

The statement “there are infinitely many prime numbers” holds iff **every structure that extends** $\mathcal{A}_2$ satisfies the formula.

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Formalization III

If the meaning of “product” is known , then we can take

Formula F_3:

$$
\begin{array}{ll}
 & \forall x\,\forall y\; Dv(x,y) \leftrightarrow \exists z\; prod(x,z) = y \\
\wedge & \forall x\; Pr(x) \leftrightarrow (\forall y\; Dv(y,x) \rightarrow (y = x \vee y = one)) \\
\rightarrow & \forall x \exists y\; Pr(y) \wedge y > x
\end{array}
$$

(the conjunction of the first two formulas implies the third)

Structure $\mathcal{A}_3$:

$$
\begin{array}{rcl}
U_{\mathcal{A}_3} & = & \mathbb{N} \\
>_{\mathcal{A}_3} & = & \{(n,m) \in \mathbb{N} \mid n > m\} \\
one_{\mathcal{A}_3} & = & 1 \\
prod_{\mathcal{A}_3}(n,m) & = & n \cdot m
\end{array}
$$

What if the meaning of “product” is not known ?

Formalization IV

If the meaning of “sum”, “successor”, “one” and “zero” is known, then we can take

Formula F_4:

$$
\begin{array}{ll}
 & \forall x \; prod(x, zero) = zero \\
\wedge & \forall x \forall y \; prod(x, succ(y)) = sum(prod(x, y), y) \\
\wedge & \forall x \; \forall y \; Dv(x, y) \leftrightarrow \exists z \; prod(x, z) = y \\
\wedge & \forall x \; Pr(x) \leftrightarrow (\forall y \; Dv(y, x) \rightarrow (y = x \vee y = one)) \\
\rightarrow & \forall x \exists y \; (Pr(y) \wedge y > x)
\end{array}
$$

Structure $\mathcal{A}_4$ only defines $>$, *sum*, *succ*, *one*, *zero*.

Observe however: *prod* is defined inductively. The definition is no longer a macro, in the sense that we cannot produce an “equivalent” formula without the symbol *prod*.

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Formalization V

Formula F_5:

$$
\begin{array}{ll}
 & \forall x\ sum(x, zero) = x \\
\wedge & \forall x \forall y\ sum(x, succ(y)) = succ(sum(x, y)) \\
\wedge & \forall x\ prod(x, zero) = zero \\
\wedge & \forall x \forall y\ prod(x, succ(y)) = sum(prod(x, y), y) \\
\wedge & \forall x\ \forall y\ (Div(x, y) \leftrightarrow \exists z\ prod(x, z) = y) \\
\wedge & \forall x\ Pri(x) \leftrightarrow (\forall y\ Div(y, x) \rightarrow (y = x \vee y = one)) \\
\rightarrow & \forall x \exists y\ (Pri(y) \wedge y > x)
\end{array}
$$

Structure $\mathcal{A}_5$ only defines $>$, *succ*, *one*, *zero*.

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\wedge & \forall x \forall y\ sum(x, succ(y)) = succ(sum(x, y)) \\
\wedge & \forall x\ prod(x, zero) = zero \\
\wedge & \forall x \forall y\ prod(x, succ(y)) = sum(prod(x, y), y) \\
\wedge & \forall x\ \forall y\ (Div(x, y) \leftrightarrow \exists z\ prod(x, z) = y) \\
\wedge & \forall x\ Pri(x) \leftrightarrow (\forall y\ Div(y, x) \rightarrow (y = x \vee y = one)) \\
\rightarrow & \forall x \exists y\ (Pri(y) \wedge y > x)
\end{array}
$$

Structure $\mathcal{A}_5$ only defines $>$, *succ*, *one*, *zero*.

What if the meaning of ‘greater than” and “one” is not known?

Formalization VI

Formula F_6:

$$
\begin{array}{ll}
 & one = succ(zero) \\
\wedge & \forall x \forall y\ x > y \leftrightarrow \exists z\ \neg(z = zero) \wedge sum(y,z) = x \\
\wedge & \forall x\ sum(x, zero) = x \\
\wedge & \forall x \forall y\ sum(x, succ(y)) = succ(sum(x,y)) \\
\wedge & \forall x\ prod(x, zero) = zero \\
\wedge & \forall x \forall y\ prod(x, succ(y)) = sum(prod(x,y), y) \\
\wedge & \forall x\ \forall y\ (Div(x,y) \leftrightarrow \exists z\ prod(x,z) = y) \\
\wedge & \forall x\ Pri(x) \leftrightarrow (\forall y\ Div(y,x) \rightarrow (y = x \vee y = one)) \\
\rightarrow & \forall x \exists y\ (Pri(y) \wedge y > x)
\end{array}
$$

Structure $\mathcal{A}_6$ only defines *succ*, *zero*.