Lecture 7

The compactness theorem

Dr Christoph Haase
University of Oxford
(with small changes by Javier Esparza)

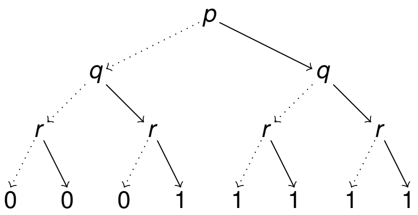
Overview

- So far: propositional logic
- Starting with the next lecture: predicate logic aka first-order logic
- Later: reduce reasoning about first-order formula
  to reasoning about infinite set of propositional formulas
- Today: reduce reasoning about infinite sets of formulas
  to reasoning about finite sets of formulas

Partial assignments

A **partial assignment** is a function $\mathcal{A} \colon D \to \{0, 1\}$, whose **domain** $D \subseteq \{p_1, p_2, \ldots\}$ is set of variables $\mathrm{dom}(\mathcal{A})$.

A partial assignment $\mathcal{A}'$ **extends** another one $\mathcal{A}$ when $\mathrm{dom}(\mathcal{A}) \subseteq \mathrm{dom}(\mathcal{A}')$ and $\mathcal{A}(p_i) = \mathcal{A}'(p_i)$ for all $p_i \in \mathrm{dom}(\mathcal{A})$.



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\mathcal{S}_2 &= \{p_1 \vee p_2,\ \neg p_2 \vee \neg p_3,\ p_3 \vee p_4,\ \neg p_4 \vee \neg p_5,\ p_5 \vee p_6,\ \ldots\} \\
\mathcal{S}_3 &= \{p_1 \vee p_2,\ \neg p_2 \vee \neg p_4,\ p_3 \vee p_6, \neg p_4 \vee \neg p_8, p_5 \vee p_{10},\ \ldots\} \\
\mathcal{S}_4 &= \{\neg p_1\} \cup \{p_n \rightarrow p_{n/2} \mid n \geq 1 \text{ and even }\} \\
&\qquad \cup \{p_n \rightarrow p_{3n+1} \mid n \geq 1 \text{ and odd }\} \cup \{p_{2^{235}-1}\}
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$\mathcal{S}_3$? $\mathcal{S}_4$?

The compactness theorem

Theorem (Compactness theorem)

A set of formulas $\mathcal{S}$ is satisfiable if and only if each finite subset of $\mathcal{S}$ is satisfiable.

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- Proof strategy: will construct good $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ with $\mathcal{A}_{n+1}$ **extending** $\mathcal{A}_n$ and $\mathrm{dom}(\mathcal{A}_n) = \{p_1, \ldots, p_n\}$

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- Base case: $\mathrm{dom}(\mathcal{A}_0) = \emptyset$

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There is assignment extending all $\mathcal{A}_n$, namely $\mathcal{A}(p_n) = \mathcal{A}_n(p_n)$
It satisfies all formulas F in S:

- if F uses variables $\{p_1, \ldots, p_n\}$, then $\mathcal{A}_n \models F$, so $\mathcal{A} \models F$

Proof of compactness theorem is **nonconstructive**
Does not give algorithm to construct a satisfying assignment, merely guarantees that one exists

Nonconstructve proofs are not really that exotic:
For every infinite sequence $a_1, a_2, a_3, \ldots$ of natural numbers there exists an index i such that $a_i \leq a_j$ for every j.

Compactness theorem: contrapositive

Compact Theorem, contrapositive: if a set of formulas is unsatisfiable, then some finite subset is already unsatisfiable

Procedure to show that infinite set of formulas is unsatisfiable:

1. enumerate $\mathcal{S} = \{F_1, F_2, \ldots\}$ by some algorithm
2. for each n, test whether $\{F_1, \ldots, F_n\}$ is unsatisfiable
3. if $\mathcal{S}$ unsatisfiable, will detect this after finite amount of time

Compactness: application

Exercise: Suppose $\{F_n \mid n \in \mathbb{N}\}$ is an infinite set of formulas such that $\{\neg F_n \mid n \in \mathbb{N}\}$ is unsatisfiable and $F_n \to F_{n+1}$ is valid for all $n \in \mathbb{N}$. Show that some F_n is valid.

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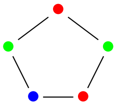
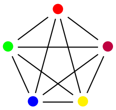
1. **Compactness**: n with $\neg F_1 \wedge \neg F_2 \wedge \ldots \wedge \neg F_n$ unsatisfiable
2. **De Morgan**: $F_1 \vee F_2 \vee \ldots \vee F_n$ is valid
3. **Resolve** $F_1 \vee F_2 \vee \ldots \vee F_n$ and $F_1 \to F_2$: see $\models F_2 \vee \ldots \vee F_n$
4. **Induction**: F_n is valid.

Graph colouring

Might as well make life even more difficult:

- Graph is k**-colourable** we can colour each vertex with $\{1, \ldots, k\}$ such that neighbours get different colours.





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- **Theorem**:
  If every finite subgraph of $\mathcal{G}$ is k-colourable, so is $\mathcal{G}$ itself.

Graph colouring: proof

Suppose all finite subgraphs of $\mathcal{G}$ are k-colourable.

- Variable $p_{v,i}$: "vertex v has colour i"
- Constraints $\mathcal{S} := \{F_v, G_v \mid v \in V\} \cup \{H_{u,v} \mid (u,v) \in E\}$:
  - Vertex v has ≥ 1 colour: $F_v := \bigvee_{i=1}^{k} p_{v,i}$
  - Vertex v has ≤ 1 colour: $G_v := \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{k} \neg p_{v,i} \vee \neg p_{v,j}$
  - Neighbours u, v different colour: $H_{u,v} := \bigwedge_{i=1}^{k} \neg p_{u,i} \vee \neg p_{v,i}$
- $\mathcal{S}$ is satisfiable iff $\mathcal{G}$ is k-colourable
- Apply Compactness Theorem

Summary: propositional logic

- Syntax
  - DNF, CNF, 2-CNF, 3-CNF
  - Horn formulas
- Semantics
  - assignments
  - truth tables
- Satisfiability: constraint problems
- Algebraic reasoning: substitution
- Polynomial-time algorithms for Horn and 2-CNF formulas, WalkSAT
- Resolution
  - Sound and complete
  - DPLL algorithm
- Compactness: nonconstructive