Lecture 6

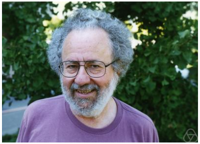
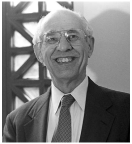
The DPLL Algorithm

Dr Christoph Haase
University of Oxford
(with small changes by Javier Esparza)

Davis–Putnam–Logemann–Loveland

DPLL algorithm:

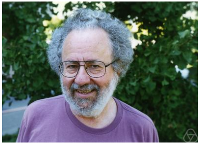
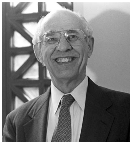
- combines search and deduction to decide satisfiability
- underlies most modern SAT solvers
- is over 50 years old

Davis–Putnam–Logemann–Loveland

DPLL algorithm:

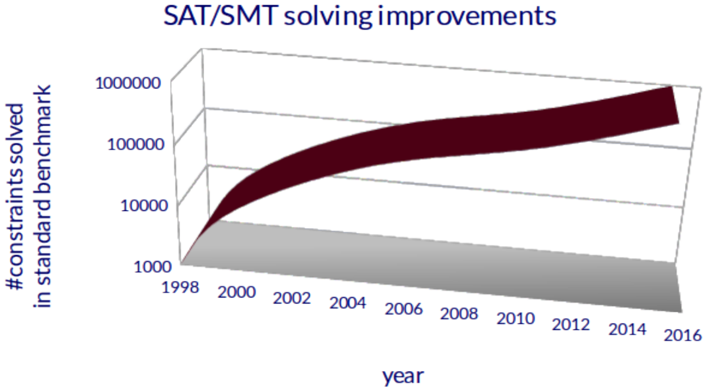
- combines search and deduction to decide satisfiability
- underlies most modern SAT solvers
- is over 50 years old

DPLL-based SAT solvers $\geq$ 1990:

- **clause learning**
- **non-chronological backtracking**
- branching heuristics
- lazy evaluation

Performance increase of SAT solvers



DPLL: idea

Depth-first search.
At every unsuccessful leaf of search tree (called **conflict**), use resolution to compute a **conflict clause**.
Add clause to formula we're deciding about.

Think of conflict clauses as "caching" previous search results, so we "learn from previous mistakes".
Conflict clauses also determine backtracking.

The DPLL algorithm

Input: CNF formula F.

1. Initialise $\mathcal{A}$ to the empty assignment
2. While there is unit clause $\{L\}$ in $F|_{\mathcal{A}}$, update $\mathcal{A} \mapsto \mathcal{A}_{[L \mapsto 1]}$
3. If $F|_{\mathcal{A}}$ contains no clauses, stop and output $\mathcal{A}$.
4. If $F|_{\mathcal{A}} \ni \square$, add new clause C to F by **learning procedure**.
   If C is the empty clause, stop and output UNSAT.
   Otherwise backtrack to highest level where C is unit clause.
   Go to Line 2.
5. Apply **decision strategy** to update $\mathcal{A} \mapsto \mathcal{A}_{[p \mapsto b]}$.
   Go to line 2.

$F|_{\mathcal{A}}$ is set of clauses obtained from deleting any clause containing true literal, and deleting from each remaining clause all false literals.

Terminology

State of algorithm is pair of CNF formula F and assignment $\mathcal{A}$.
Successful state when $\mathcal{A} \models F$. **Conflict state** when $\mathcal{A} \not\models F$.

Terminology

State of algorithm is pair of CNF formula F and assignment $\mathcal{A}$.
Successful state when $\mathcal{A} \models F$. **Conflict state** when $\mathcal{A} \not\models F$.

- Each assignment $p_i \mapsto b_i$ classifies as **decision assignment** or **implied assignment**.

Terminology

State of algorithm is pair of CNF formula F and assignment $\mathcal{A}$.
Successful state when $\mathcal{A} \models F$. **Conflict state** when $\mathcal{A} \not\models F$.

- Each assignment $p_i \mapsto b_i$ classifies as **decision assignment** or **implied assignment**.
- p_i in a decision assignment $p_i \mapsto b_i$ is **decision variable.**

Terminology

State of algorithm is pair of CNF formula F and assignment $\mathcal{A}$.
Successful state when $\mathcal{A} \models F$. **Conflict state** when $\mathcal{A} \not\models F$.

- Each assignment $p_i \mapsto b_i$ classifies as **decision assignment** or **implied assignment**.
- p_i in a decision assignment $p_i \mapsto b_i$ is **decision variable.**
- Denote by $p_i \overset{C}{\mapsto} b_i$ an implied assignment arising through **unit propagation** on clause C.

Terminology

State of algorithm is pair of CNF formula F and assignment $\mathcal{A}$.
Successful state when $\mathcal{A} \models F$. **Conflict state** when $\mathcal{A} \not\models F$.

- Each assignment $p_i \mapsto b_i$ classifies as **decision assignment** or **implied assignment**.
- p_i in a decision assignment $p_i \mapsto b_i$ is **decision variable**.
- Denote by $p_i \overset{C}{\mapsto} b_i$ an implied assignment arising through **unit propagation** on clause C.
- **Decision level** of assignment $p_i \mapsto b_i$ in a given state $\mathcal{A}$ is number of decision assignments in $\mathcal{A}$ that precede $p_i \mapsto b_i$.

Terminology

State of algorithm is pair of CNF formula F and assignment $\mathcal{A}$.
Successful state when $\mathcal{A} \models F$. **Conflict state** when $\mathcal{A} \not\models F$.

- Each assignment $p_i \mapsto b_i$ classifies as **decision assignment** or **implied assignment**.
- p_i in a decision assignment $p_i \mapsto b_i$ is **decision variable.**
- Denote by $p_i \overset{C}{\mapsto} b_i$ an implied assignment arising through **unit propagation** on clause C.
- **Decision level** of assignment $p_i \mapsto b_i$ in a given state $\mathcal{A}$ is number of decision assignments in $\mathcal{A}$ that precede $p_i \mapsto b_i$.

(Note: conflict state if $F|_{\mathcal{A}} \ni \square$, successful state if $F|_{\mathcal{A}} = \emptyset$)

Unit propagation

Unit propagation: the while loop in line 2 updates assignment $L \mapsto 1$ whenever there is unit clause $\{L\} \in F|_{\mathcal{A}}$.

Unit propagation

Unit propagation: the while loop in line 2 updates assignment $L \mapsto 1$ whenever there is unit clause $\{L\} \in F|_{\mathcal{A}}$.

Example: start with set of clauses $F = \{C_1, \ldots, C_5\}$, where

$$
\begin{aligned}
C_1 &= \{\neg p_1, \neg p_4, p_5\} \\
C_2 &= \{\neg p_1, p_6, \neg p_5\} \\
C_3 &= \{\neg p_1, \neg p_6, p_7\} \\
C_4 &= \{\neg p_1, \neg p_7, \neg p_5\} \\
C_5 &= \{p_1, p_4, p_6\}
\end{aligned}
$$

Say current assignment is $(p_1 \mapsto 1, p_2 \mapsto 0, p_3 \mapsto 0, p_4 \mapsto 1)$.
Notice $F|_{\mathcal{A}}$ contains unit clause $\{p_5\}$.

Unit propagation further generates $(p_5 \overset{C_1}{\mapsto} 1, p_6 \overset{C_2}{\mapsto} 1, p_7 \overset{C_3}{\mapsto} 1)$. This leads to a conflict, with C_4 being made false.

Conflict analysis

After unit propagation:

- If not in conflict nor successful, make decision (line 5)
- If in conflict, **learned clause** is added (line 4)

Conflict analysis

After unit propagation:

- If not in conflict nor successful, make decision (line 5)
- If in conflict, **learned clause** is added (line 4)

Learned clause desiderata: If unit propagation from state $(F, \mathcal{A})$ leads to conflict, clause C is learned such that:

Conflict analysis

After unit propagation:

- If not in conflict nor successful, make decision (line 5)
- If in conflict, **learned clause** is added (line 4)

Learned clause desiderata: If unit propagation from state $(F, \mathcal{A})$ leads to conflict, clause C is learned such that:

1. $F \equiv F \cup \{C\}$

Conflict analysis

After unit propagation:

- If not in conflict nor successful, make decision (line 5)
- If in conflict, **learned clause** is added (line 4)

Learned clause desiderata: If unit propagation from state $(F, \mathcal{A})$ leads to conflict, clause C is learned such that:

1. $F \equiv F \cup \{C\}$
2. C is **conflict clause**: each literal is made false by $\mathcal{A}$

Conflict analysis

After unit propagation:

- If not in conflict nor successful, make decision (line 5)
- If in conflict, **learned clause** is added (line 4)

Learned clause desiderata: If unit propagation from state $(F, \mathcal{A})$ leads to conflict, clause C is learned such that:

1. $F \equiv F \cup \{C\}$
2. C is **conflict clause**: each literal is made false by $\mathcal{A}$
3. C mentions only decision variables in $\mathcal{A}$

Clause learning

Suppose $\mathcal{A} = (p_1 \mapsto b_1, \ldots, p_k \mapsto b_k)$ leads to conflict.
Find associated clauses $A_1, \ldots, A_{k+1}$ by backward induction:

Clause learning

Suppose $\mathcal{A} = (p_1 \mapsto b_1, \ldots, p_k \mapsto b_k)$ leads to conflict.
Find associated clauses $A_1, \ldots, A_{k+1}$ by backward induction:

1. Take any conflict clause under $\mathcal{A}$ as A_{k+1}

Clause learning

Suppose $\mathcal{A} = (p_1 \mapsto b_1, \ldots, p_k \mapsto b_k)$ leads to conflict.
Find associated clauses $A_1, \ldots, A_{k+1}$ by backward induction:

1. Take any conflict clause under $\mathcal{A}$ as A_{k+1}
2. If $p_i \mapsto b_i$ is decision assignment or p_i not mentioned in A_{i+1}, set $A_i = A_{i+1}$

Clause learning

Suppose $\mathcal{A} = (p_1 \mapsto b_1, \ldots, p_k \mapsto b_k)$ leads to conflict.
Find associated clauses $A_1, \ldots, A_{k+1}$ by backward induction:

1. Take any conflict clause under $\mathcal{A}$ as A_{k+1}
2. If $p_i \mapsto b_i$ is decision assignment or p_i not mentioned in A_{i+1}, set $A_i = A_{i+1}$
3. If $p_i \overset{C_i}{\mapsto} b_i$ is implied assignment and p_i mentioned in A_{i+1}, define A_i to be resolvent of A_{i+1} and C_i with respect to p_i

Clause learning

Suppose $\mathcal{A} = (p_1 \mapsto b_1, \ldots, p_k \mapsto b_k)$ leads to conflict.
Find associated clauses $A_1, \ldots, A_{k+1}$ by backward induction:

1. Take any conflict clause under $\mathcal{A}$ as A_{k+1}
2. If $p_i \mapsto b_i$ is decision assignment or p_i not mentioned in A_{i+1}, set $A_i = A_{i+1}$
3. If $p_i \overset{C_i}{\mapsto} b_i$ is implied assignment and p_i mentioned in A_{i+1}, define A_i to be resolvent of A_{i+1} and C_i with respect to p_i

The final clause A_1 is the **learned clause**

Clause learning: example

In conflict of above example, learning generates clauses

$$
\begin{aligned}
A_8 &:= \{\neg p_1, \neg p_7, \neg p_5\} && \text{(clause } C_4\text{)} \\
A_7 &:= \{\neg p_1, \neg p_5, \neg p_6\} && \text{(resolve } A_8,\ C_3\text{)} \\
A_6 &:= \{\neg p_1, \neg p_5\} && \text{(resolve } A_7,\ C_2\text{)} \\
A_5 &:= \{\neg p_1, \neg p_4\} && \text{(resolve } A_6,\ C_1\text{)} \\
&\ \vdots \\
A_1 &:= \{\neg p_1, \neg p_4\}
\end{aligned}
$$

Clause learning: example

In conflict of above example, learning generates clauses

$$
\begin{aligned}
A_8 &:= \{\neg p_1, \neg p_7, \neg p_5\} && \text{(clause } C_4\text{)} \\
A_7 &:= \{\neg p_1, \neg p_5, \neg p_6\} && \text{(resolve } A_8, C_3\text{)} \\
A_6 &:= \{\neg p_1, \neg p_5\} && \text{(resolve } A_7, C_2\text{)} \\
A_5 &:= \{\neg p_1, \neg p_4\} && \text{(resolve } A_6, C_1\text{)} \\
&\vdots \\
A_1 &:= \{\neg p_1, \neg p_4\}
\end{aligned}
$$

Learned clause A_1 is conflict clause with only decision variables, including top-level one p_4. Intuitively:

- record that conflict arose from decision to make p_1, p_4 true
- adding A_1 makes assignments validating p_1, p_4 unreachable
- backtrack to highest level where A_1 is unit clause ($p_1 \mapsto 1$), unit propagation leads to $p_4 \mapsto 0$.

Clause learning

Proposition: this policy fulfills the desiderata

Proof sketch: Observation: If $p_i \stackrel{C_i}{\mapsto} b_i$, then the only literal of C_i true under $\mathcal{A}$ is the literal for p_i (that is, C_i contains either p_i or $\neg p_i$, and b_i is chosen to make the literal true).

Clause learning

Proposition: this policy fulfills the desiderata

Proof sketch: Observation: If $p_i \stackrel{C_i}{\mapsto} b_i$, then the only literal of C_i true under $\mathcal{A}$ is the literal for p_i (that is, C_i contains either p_i or $\neg p_i$, and b_i is chosen to make the literal true).

1. $F \equiv F \cup \{C\}$
   Because C is obtained from clauses of F through resolution steps.

Clause learning

Proposition: this policy fulfills the desiderata

Proof sketch: Observation: If $p_i \overset{C_i}{\mapsto} b_i$, then the only literal of C_i true under $\mathcal{A}$ is the literal for p_i (that is, C_i contains either p_i or $\neg p_i$, and b_i is chosen to make the literal true).

1. $F \equiv F \cup \{C\}$
   Because C is obtained from clauses of F through resolution steps.
2. C is **conflict clause**: each literal is made false by $\mathcal{A}$.
   We show by induction that it holds for $A_{k+1}, A_k, A_{k-1} \cdots A_1 = C$.
   It holds for A_{k+1} by definition.
   If it holds for A_{i+1} and $A_i = A_{i+1}$, then obviously it holds for A_i.
   If it holds for A_{i+1} and $A_i \neq A_{i+1}$, then A_i is the result of resolving A_{i+1} and C_i. By the observation, all literals of A_i are made false by $\mathcal{A}$.

Clause learning

Proposition: this policy fulfills the desiderata

Proof sketch: Observation: If $p_i \stackrel{C_i}{\mapsto} b_i$, then the only literal of C_i true under $\mathcal{A}$ is the literal for p_i (that is, C_i contains either p_i or $\neg p_i$, and b_i is chosen to make the literal true).

1. $F \equiv F \cup \{C\}$
   Because C is obtained from clauses of F through resolution steps.
2. C is **conflict clause**: each literal is made false by $\mathcal{A}$.
   We show by induction that it holds for $A_{k+1}, A_k, A_{k-1} \cdots A_1 = C$.
   It holds for A_{k+1} by definition.
   If it holds for A_{i+1} and $A_i = A_{i+1}$, then obviously it holds for A_i.
   If it holds for A_{i+1} and $A_i \neq A_{i+1}$, then A_i is the result of resolving A_{i+1} and C_i. By the observation, all literals of A_i are made false by $\mathcal{A}$.
3. C mentions only decision variables in $\mathcal{A}$.
   Because every other variable, say p_i, dissapears after resolving with A_{i+1} w.r.t. p_i. Indeed, since $\mathcal{A}$ makes A_{i+1} false, by the observation p_i has opposite signs in A_{i+1} and C_i.

Example: 4 queens

Problem: place 4 non-attacking queens on a 4x4 chess board

Example: 4 queens

Problem: place 4 non-attacking queens on a 4x4 chess board
Variable p_{ij} models: there is a queen in square (i, j)

- ≥ 1 in each row: $\bigwedge_{i=1}^{4} \bigvee_{j=1}^{4} p_{ij}$
- ≤ 1 in each row: $\bigwedge_{i=1}^{4} \bigwedge_{j \neq j'=1}^{4} \neg p_{ij} \vee \neg p_{ij'}$
- ≤ 1 in each column: $\bigwedge_{j=1}^{4} \bigwedge_{i \neq i'=1}^{4} \neg p_{ij} \vee \neg p_{i'j}$
- ≤ 1 on each diagonal: $\bigwedge_{i,j=1}^{4} \bigvee_{k} \neg p_{i-k,i+k} \vee \neg p_{i+k,j+k}$

Total number of clauses: 4 + 24 + 24 + 28 = 80

DPLL: 4 queens

Running the DPLL algorithm:

- Start with $p_{11} \mapsto 1$
  delete $\{p_{11}, p_{12}, p_{13}, p_{14}\}$, delete $\neg p_{11}$: 9 new unit clauses
  unit propagation: deletes 65 clauses!

DPLL: 4 queens

Running the DPLL algorithm:

- Start with $p_{11} \mapsto 1$
  delete $\{p_{11}, p_{12}, p_{13}, p_{14}\}$, delete $\neg p_{11}$: 9 new unit clauses
  unit propagation: deletes 65 clauses!

- Set $p_{23} \mapsto 1$
  4 new unit clauses: $\{\neg p_{24}\}, \{\neg p_{43}\}, \{\neg p_{32}\}, \{\neg p_{34}\}$
  unit propagation of $\{\neg p_{34}\}$: UNSAT

DPLL: 4 queens

Running the DPLL algorithm:

- Start with $p_{11} \mapsto 1$
  delete $\{p_{11}, p_{12}, p_{13}, p_{14}\}$, delete $\neg p_{11}$: 9 new unit clauses
  unit propagation: deletes 65 clauses!

- Set $p_{23} \mapsto 1$
  4 new unit clauses: $\{\neg p_{24}\}, \{\neg p_{43}\}, \{\neg p_{32}\}, \{\neg p_{34}\}$
  unit propagation of $\{\neg p_{34}\}$: UNSAT

  fixing only two literals collapsed from 80 clauses to 1
  ruled out 2^{14} of 2^{16} possible assignments!

- Backtrack: $p_{11} \mapsto 0$, $p_{12} \mapsto 1$
  delete $\{\neg p_{12}\}$: 9 new unit clauses
  unit propagation: leaves only 1 clause $\{p_{43}\}$!

DPLL: 4 queens

Running the DPLL algorithm:

- Start with $p_{11} \mapsto 1$
  delete $\{p_{11}, p_{12}, p_{13}, p_{14}\}$, delete $\neg p_{11}$: 9 new unit clauses
  unit propagation: deletes 65 clauses!

- Set $p_{23} \mapsto 1$
  4 new unit clauses: $\{\neg p_{24}\}, \{\neg p_{43}\}, \{\neg p_{32}\}, \{\neg p_{34}\}$
  unit propagation of $\{\neg p_{34}\}$: UNSAT

  fixing only two literals collapsed from 80 clauses to 1
  ruled out 2^{14} of 2^{16} possible assignments!

- Backtrack: $p_{11} \mapsto 0$, $p_{12} \mapsto 1$
  delete $\{\neg p_{12}\}$: 9 new unit clauses
  unit propagation: leaves only 1 clause $\{p_{43}\}$!

- Answer: $p_{12}, p_{24}, p_{31}, p_{43} \mapsto 1$

Summary

- Resolution:
  - very simple sound and complete proof calculus
  - basis for type unification
- DPLL algorithm
  - improves resolution with clause learning and backtracking
  - very efficient basis for modern SAT solvers