Lecture 2

Propositional logic

syntax and semantics, the satisfiability problem, constraint problems

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Agenda

1. Propositional Logic
2. Syntax and semantics of propositional logic
3. Encoding constraint problems into satisfiability problems

Propositional logic

- Formulas of propositional logic (aka propositions) are built of smaller formulas using *connectives* like *and*, *or*, *not*, *implies*, and others.
- The smallest formulas are *propositional variables*, aka *atomic propositions* or *atoms*, which can be instantiated with statements which are either true or false.
- A prime concern: *given a compound formula, determine which truth values of its atoms make it true*.

An example

- Atomic propositions:

Atoms	Instance
a	“Alice is an architect”
b	“Bob is a builder”
c	“Charlie is a cook”

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- These entail that Alice is an architect: if the above three propositions are all true then a *must* also be true ($\{\neg c, a \vee b, b \to c\} \models a$).
- The correctness of this entailment is *independent* of the instantiation of the atomic propositions!

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Syntax of propositional logic

Definition (Syntax of propositional logic)

Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of **propositional variables**. **Formulas** of propositional logic are inductively defined as follows:

1. *true* and *false* are formulas.
2. Every propositional variable x_i is a formula.
3. If F is a formula, then $\neg F$ is a formula.
4. If F and G are formulas, then $(F \wedge G)$ and $(F \vee G)$ are formulas.

Additional notation

- We often write x, y, z or p to denote propositional variables.
- We call $\neg F$ the **negation** of F.
- Given formulas F and G, $(F \wedge G)$ is the **conjunction** of F and G, and $(F \vee G)$ is the **disjunction** of F and G.
- We call $\neg$, $\wedge$ and $\vee$ **logical connectives**.
- We denote by $\mathcal{F}(X)$ the **set of all formulas** built from propositional variables in X.

Derived connectives

- **Implication**: $(F_1 \rightarrow F_2) := (\neg F_1 \vee F_2)$
- **Bi-implication**: $(F_1 \leftrightarrow F_2) := (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$
- **Exclusive Or**: $(F_1 \oplus F_2) := (F_1 \wedge \neg F_2) \vee (\neg F_1 \wedge F_2)$
- **Indexed Conjunction**: $\bigwedge_{i=1}^{n} F_i := (\cdots((F_1 \wedge F_2) \wedge F_3) \wedge \cdots \wedge F_n)$
- **Indexed Disjunction**: $\bigvee_{i=1}^{n} F_i := (\cdots((F_1 \vee F_2) \vee F_3) \vee \cdots \vee F_n)$
- Note on bracketing:
  - We usually drop outer brackets
  - Operator precedences: $\leftrightarrow$ and $\rightarrow$ bind weaker than $\wedge$ and $\vee$, which bind weaker than $\neg$.
  - Example: $\neg x \wedge y \rightarrow z$ means $((\neg x \wedge y) \rightarrow z)$

Syntax trees

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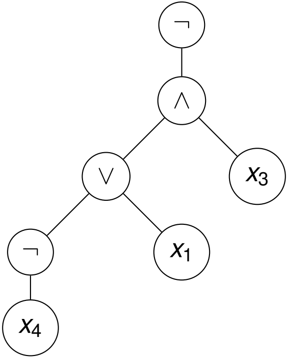
Example: syntax tree of $\neg((\neg x_4 \vee x_1) \wedge x_3)$:



Inductive definitions

Inductive definition of formulas allows us to define functions on formulas by **structural induction**, by defining the function

- For the base cases *true*, *false* and x_i, and
- For the induction steps $\neg F$, $F \wedge G$ and $F \vee G$.

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Example

The function $sub\colon \mathcal{F}(X) \to 2^{\mathcal{F}(X)}$ returning the set of all subformulas of a given formula can be defined by:

- $sub(true) = \{true\}$, $sub(false) = \{false\}$
- $sub(x) = \{x\}$ for all $x \in X$
- $sub(\neg F) = \{\neg F\} \cup sub(F)$
- $sub(F \wedge G) = \{F \wedge G\} \cup sub(F) \cup sub(G)$
- $sub(F \vee G) = \{F \vee G\} \cup sub(F) \cup sub(G)$

Syntax vs semantics

The *syntax* tells us how we write something down, the *semantics* what it means:

- syntax: some formal *language*
- semantics: some mathematical *object*
- our syntax: propositional formulas
- our semantics: **truth tables**

Semantics of propositional logic

Definition

An **assignment** is a function $\mathcal{A}\colon X \to \{0,1\}$ that induces an assignment $\hat{\mathcal{A}}\colon \mathcal{F}(X) \to \{0,1\}$ by structural induction as follows:

1. $\hat{\mathcal{A}}(\mathit{false}) = 0$, $\hat{\mathcal{A}}(\mathit{true}) = 1$
2. For every $x \in X$, $\hat{\mathcal{A}}(x) := \mathcal{A}(x)$
3. $\hat{\mathcal{A}}(\neg F) := \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 0 \\ 0 & \text{otherwise} \end{cases}$
4. $\hat{\mathcal{A}}((F \wedge G)) := \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ and } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases}$
5. $\hat{\mathcal{A}}((F \vee G)) := \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(F) = 1 \text{ or } \hat{\mathcal{A}}(G) = 1 \\ 0 & \text{otherwise} \end{cases}$

Semantics of propositional logic

- The semantics of a formula F is the function that maps each assignment $\mathcal{A}\colon X \to \{0,1\}$ to the truth value $\hat{\mathcal{A}}(F)$.
- Let $Y \subseteq X$ be the set of variables occurring in F. $\hat{\mathcal{A}}(F)$ is completely determined by the values assigned by $\mathcal{A}$ to the variables of Y.
- With a slight abuse of language, we also say that the semantics of F is the function that maps each restricted assignment $\mathcal{A}'\colon Y \to \{0,1\}$ to the truth value $\widehat{\mathcal{A}'}(F)$.
- Observe that X is infinite, but Y is finite. So there is an uncountable infinity of assignments, but only $2^{|Y|}$ restricted assignments.

Semantics of propositional logic

Example

Let $F = (x \wedge \neg y) \vee z$ and $\mathcal{A}$ be an assignment such that $\mathcal{A}(x) = 1$ and $\mathcal{A}(y) = \mathcal{A}(z) = 0$. Then F evaluates to true under $\mathcal{A}$, since

$$\begin{aligned}
\hat{\mathcal{A}}(F) &= \begin{cases} 1 & \text{if } \hat{\mathcal{A}}((x \wedge \neg y)) = 1 \text{ or } \hat{\mathcal{A}}(z) = 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \hat{\mathcal{A}}((x \wedge \neg y)) = 1 \text{ (since } \mathcal{A}(z) = 0) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \hat{\mathcal{A}}(x) = 1 \text{ and } \hat{\mathcal{A}}(\neg y) = 1 \\ 0 & \text{otherwise} \end{cases} \\
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\end{aligned}$$

Subsequently we will not write the hat on top of $\mathcal{A}$.

Semantics via truth tables

Example

The semantics of logical connectives via **truth tables**:

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}(F \wedge G)$
0	0	0
1	0	0
0	1	0
1	1	1

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}(F \vee G)$
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$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}(F \rightarrow G)$
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1	0	0
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$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}(F \oplus G)$
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Formalising natural language: an example

A device consists of a thermostat, a pump, and a warning light.
Suppose we are told the following four facts about the pump:

- The thermostat or the pump (or both) are broken.
- If the thermostat is broken then the pump is also broken.
- If the pump is broken and the warning light is on then the thermostat is not broken.
- The warning light is on.

Is it possible for all four to be true at the same time?

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Is it possible for all four to be true at the same time?
In a propositional formula:

$$F := (t \vee p) \wedge (t \rightarrow p) \wedge ((p \wedge w) \rightarrow \neg t) \wedge w$$

Truth table

$$F := (t \lor p) \land (t \to p) \land ((p \land w) \to \neg t) \land w$$

t	p	w	f
0	0	0	0
0	0	1	0
0	1	0	0
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There is a unique assignment that makes F true. We can think of each assignment as describing a *possible world*, and there is only one world in which F is true.

Models, satisfiability and validity

Definition

Let $F \in \mathcal{F}(X)$ and $\mathcal{A}\colon X \to \{0, 1\}$ be an assignment.

1. If $\mathcal{A}(F) = 1$ then we write $\mathcal{A} \models F$ ("F *holds under* $\mathcal{A}$", or "$\mathcal{A}$ is a **model** of F".)
2. If F has at least one model then F is **satisfiable**, otherwise F is **unsatisfiable**.
3. If F holds under any assignment $\mathcal{A}\colon X \to \{0, 1\}$ then F is called **valid** or a **tautology**, written $\models F$.

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Definition

Given $F \in \mathcal{F}(X)$, the **Boolean satisfiability problem (SAT)** is to decide whether F is satisfiable.

Models, satisfiability and validity

Example

The subsequent first two tautologies are known as the *distributive laws*, the last two as *De Morgan's laws*:

$$\models (F \vee (G \wedge H)) \leftrightarrow ((F \vee G) \wedge (F \vee H))$$
$$\models (F \wedge (G \vee H)) \leftrightarrow ((F \wedge G) \vee (F \wedge H))$$
$$\models \neg(F \wedge G) \leftrightarrow \neg F \vee \neg G$$
$$\models \neg(F \vee G) \leftrightarrow \neg F \wedge \neg G.$$

Entailment and equivalence

Definition (Entailment)

A formula G is a **consequence** of (or is **entailed** by) a set of formulas $\mathcal{S}$ if every assignment that satisfies each formula in $\mathcal{S}$ also satisfies G. In this case we write $\mathcal{S} \models G$.

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Definition (Equivalence)

Two formulas F and G are said to be **logically equivalent** if $\mathcal{A}(F) = \mathcal{A}(G)$ for every assignment $\mathcal{A}$. We write $F \equiv G$ to denote that F and G are equivalent.

1. Propositional Logic
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Sudoku

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

Sudoku

For each $i, j, k \in \{1, \ldots, 9\}$ we have a proposition $x_{i,j,k}$ expressing that *grid position* i, j *contains number* k. Build formula F as the conjunction of the following *constraints*:

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- Each number appears in each row and in each column:

$$F_1 := \bigwedge_{i=1}^{9} \bigwedge_{k=1}^{9} \bigvee_{j=1}^{9} x_{i,j,k} \qquad F_2 := \bigwedge_{j=1}^{9} \bigwedge_{k=1}^{9} \bigvee_{i=1}^{9} x_{i,j,k}$$

- Each number appears in each 3×3 block:

$$F_3 := \bigwedge_{k=1}^{9} \bigwedge_{u=0}^{2} \bigwedge_{v=0}^{2} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} x_{3u+i,3v+j,k}$$

- No square contains two numbers:

$$F_4 := \bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{1 \le k < k' \le 9} \neg(x_{i,j,k} \wedge x_{i,j,k'}) \,.$$

Sudoku

- Certain numbers appear in certain positions: we assert

$$F_5 := x_{2,1,2} \wedge x_{1,2,8} \wedge x_{2,3,3} \wedge \ldots \wedge x_{8,9,6} .$$

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
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		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

Sudoku

- Missing constraints? What about: no number appears twice in the same row?

$$F_6 := \bigwedge_{i=1}^{9} \bigwedge_{k=1}^{9} \bigwedge_{1 \leq j < j' < 9} \neg(x_{i,j,k} \wedge x_{i,j',k})$$

- Entailed by the existing formulas: adding F_6 as an extra constraint would not change the set of satisfying assignments.

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- But adding logically redundant constraints may help a computer search for a satisfying assignment.

- The number of variables $x_{i,j,k}$ is $9^3 = 729$. Thus a truth table for the corresponding formula would have $2^{729} > 10^{200}$ lines! Nevertheless a modern SAT-solver can find a satisfying assignment in milliseconds.

Hamiltonian path

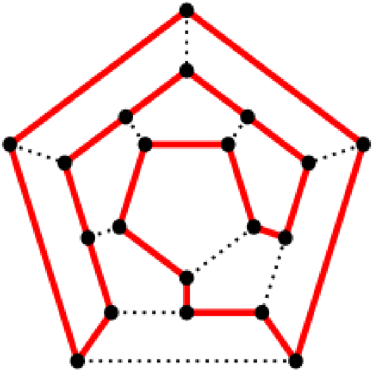

Figure: Example of a Hamiltonian path in an undirected graph.

Hamiltonian path

Given an undirected graph $G = (V, E)$ such that $E \subseteq V \times V$ is symmetric, for each vertex $i, j \in \{1, \ldots, n\}$ we have a proposition $x_{i,j}$ expressing that *vertex i is the jth vertex in the Hamiltonian path*. Build formula F as the conjunction of the following *constraints*:

- Each vertex is visited precisely once:

$$F_1 := \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} x_{i,j} \qquad F_2 := \bigwedge_{i=1}^{n} \bigwedge_{1 \leq j \neq k \leq n} \neg(x_{i,j} \wedge x_{i,k}) \wedge \neg(x_{j,i} \wedge x_{k,i})$$

- The path goes along edges:

$$F_4 := \bigwedge_{i=1}^{n} \bigwedge_{k=1}^{n} \bigwedge_{j=1}^{n-1} x_{i,j} \wedge x_{k,j+1} \rightarrow e_{i,k}$$

$$F_5 := \bigwedge_{(i,j) \in E} e_{i,j} \wedge \bigwedge_{(i,j) \notin E} \neg e_{i,j}$$

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- Can do better for special formula classes: Horn formulas, 2-CNF formulas, XOR-clauses, ...
- Reductions of combinatorial problems to SAT should run in polynomial-time!