Lecture 16

Definability in first-order logic

elementary equivalences, isomorphisms, separation of structures by games

Logic and Proof
5 June 2019

Dr Christoph Haase
University of Oxford

Debt from last lecture

Theorem (Cantor)

Any two countable unbounded dense linear orders are isomorphic.

Figure: Georg Cantor (1845 – 1918)

Today: Expressiveness of first-order logic

- Definability in a fixed structure
- Axiomatisations of classes of structures
- Work with relational structures but results hold w.l.o.g. for arbitrary structures

Definability of relations in a structure

Want know whether a relation is definable in a structure:

- Can define "$<$" in the stucture $(\mathbb{N}, 0, 1, +, =)$:

$$L(x, y) \equiv \exists z\, x + z = y \wedge \neg(z = 0).$$

Definability of relations in a structure

Want know whether a relation is definable in a structure:

- Can define "$<$" in the stucture $(\mathbb{N}, 0, 1, +, =)$:

$$L(x, y) \equiv \exists z\, x + z = y \wedge \neg(z = 0).$$

- What if we consider $(\mathbb{Z}, 0, 1, +, =)$?

Definability of relations in a structure

Want know whether a relation is definable in a structure:

- Can define "$<$" in the stucture $(\mathbb{N}, 0, 1, +, =)$:

$$L(x, y) \equiv \exists z\, x + z = y \wedge \neg(z = 0).$$

- What if we consider $(\mathbb{Z}, 0, 1, +, =)$?

Definition

Let $\mathcal{A}$ be a σ-structure. Then $R \subseteq U_{\mathcal{A}}^n$ is **elementary definable in $\mathcal{A}$** if there is $F(x_1, \ldots, x_n)$ such that

$$R = \{(a_1, \ldots, a_n) : \mathcal{A}_{[x_1 \mapsto a_1]\cdots[x_n \mapsto a_n]} \models F\}$$

Isomorphisms between structures

Definition

Two relational σ-strucutres $\mathcal{A}, \mathcal{B}$ are **isomorphic** if there exists a bijection $h\colon U_{\mathcal{A}} \to U_{\mathcal{B}}$ such that for every predicate symbol P

$$(a_1, \ldots, a_n) \in P_{\mathcal{A}} \Leftrightarrow (h(a_1), \ldots, h(a_n)) \in P_{\mathcal{B}}$$

Isomorphisms between structures

Definition

Two relational σ-strucutres $\mathcal{A}, \mathcal{B}$ are **isomorphic** if there exists a bijection $h\colon U_{\mathcal{A}} \to U_{\mathcal{B}}$ such that for every predicate symbol P

$$(a_1, \ldots, a_n) \in P_{\mathcal{A}} \Leftrightarrow (h(a_1), \ldots, h(a_n)) \in P_{\mathcal{B}}$$

- Isomorphism $h\colon U_{\mathcal{A}} \to U_{\mathcal{A}}$ is an **automorphism**
- Automorphisms form a group
- Want that first-order logic cannot distinguish between isomorphic structures

The isomorphism lemma

Lemma

Let $h\colon U_{\mathcal{A}} \to U_{\mathcal{B}}$ be an isomporphism. Then for all σ-formulas $F(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in U_{\mathcal{A}}$, we have

$$\mathcal{A}_{[x_1 \mapsto a_1]\cdots[x_n \mapsto a_n]} \models F \qquad \textit{iff} \qquad \mathcal{B}_{[x_1 \mapsto h(a_1)]\cdots[x_n \mapsto h(a_n)]} \models F$$

The isomorphism lemma

Lemma

Let $h\colon U_{\mathcal{A}} \to U_{\mathcal{B}}$ be an isomporphism. Then for all σ-formulas $F(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in U_{\mathcal{A}}$, we have

$$\mathcal{A}_{[x_1 \mapsto a_1]\cdots[x_n \mapsto a_n]} \models F \qquad \textit{iff} \qquad \mathcal{B}_{[x_1 \mapsto h(a_1)]\cdots[x_n \mapsto h(a_n)]} \models F$$

Proof.

Structural induction on F. □

Proving non-definability of a relation

Can use isomoprhism lemma to show that relation is not definable in a given structure

Proving non-definability of a relation

Can use isomoprhism lemma to show that relation is not definable in a given structure

Lemma

If $h\colon U_{\mathcal{A}} \to U_{\mathcal{A}}$ is an automorphism, then h is also an automorphism when adding a relation R to $\mathcal{A}$ that is elementary definable in $\mathcal{A}$.

Proof.

Let $F(x_1, \ldots, x_n)$ define R, by the isomorphism lemma we have $\mathcal{A}_{[x_1 \mapsto a_1] \cdots [x_n \mapsto a_n]} \models F$ iff $\mathcal{A}_{[x_1 \mapsto h(a_1)] \cdots [x_n \mapsto h(a_n)]} \models F$. Hence $R = h(R)$. □

Proving non-definability of a relation

Can use isomoprhism lemma to show that relation is not definable in a given structure

Lemma

If $h\colon U_{\mathcal{A}} \to U_{\mathcal{A}}$ is an automorphism, then h is also an automorphism when adding a relation R to $\mathcal{A}$ that is elementary definable in $\mathcal{A}$.

Proof.

Let $F(x_1, \ldots, x_n)$ define R, by the isomorphism lemma we have $\mathcal{A}_{[x_1 \mapsto a_1] \cdots [x_n \mapsto a_n]} \models F$ iff $\mathcal{A}_{[x_1 \mapsto h(a_1)] \cdots [x_n \mapsto h(a_n)]} \models F$. Hence $R = h(R)$. □

Example

Observe that $h\colon x \mapsto -x$ is an automorphism in $\mathcal{A} = (\mathbb{Z}, 0, 1, +, =)$. Clearly, we do not have $x < y$ iff $-x < -y$. Hence, the order relation "$<$" is not definable in $\mathcal{A}$.

Elementary equivalent structures

Definition

We call $\mathcal{A}$ and $\mathcal{B}$ **elementary equivalent** ($\mathcal{A} \equiv \mathcal{B}$) if for all sentences F we have

$$\mathcal{A} \models F \iff \mathcal{B} \models F.$$

Elementary equivalent structures

Definition

We call $\mathcal{A}$ and $\mathcal{B}$ **elementary equivalent** ($\mathcal{A} \equiv \mathcal{B}$) if for all sentences F we have

$$\mathcal{A} \models F \iff \mathcal{B} \models F.$$

Call $\mathcal{A}$ and $\mathcal{B}$ m**-equivalent** ($\mathcal{A} \equiv_m \mathcal{B}$) if for all F with quantifier-depth at most m, we have

$$\mathcal{A} \models F \iff \mathcal{B} \models F.$$

Elementary equivalent structures

Definition

We call $\mathcal{A}$ and $\mathcal{B}$ **elementary equivalent** ($\mathcal{A} \equiv \mathcal{B}$) if for all sentences F we have

$$\mathcal{A} \models F \iff \mathcal{B} \models F.$$

Call $\mathcal{A}$ and $\mathcal{B}$ m**-equivalent** ($\mathcal{A} \equiv_m \mathcal{B}$) if for all F with quantifier-depth at most m, we have

$$\mathcal{A} \models F \iff \mathcal{B} \models F.$$

Isomorphic structures are, in particular, elementary equivalent.

Cantor's theorem

Theorem (Cantor)

Any two countable unbounded dense linear orders are isomorphic.

Proof.

Since $\mathcal{A}$ and $\mathcal{B}$ are countable, enumerate their elements:

$$a_1 \; a_2 \; a_3 \; \cdots \text{ and } b_1 \; b_2 \; b_3 \; \cdots$$

Inductively define new enumerations

$$a'_1 \; a'_2 \; a'_3 \; \cdots \text{ and } b'_1 \; b'_2 \; b'_3 \; \cdots$$

such that $a'_i < a'_j$ iff $b'_i < b'_j$. The isomorphism h is $h(a'_i) := b'_i$ for all $i > 0$.

Cantor's theorem

Theorem (Cantor)

Any two countable unbounded dense linear orders are isomorphic.

Proof.

Since $\mathcal{A}$ and $\mathcal{B}$ are countable, enumerate their elements:

$$a_1 \; a_2 \; a_3 \; \cdots \text{ and } b_1 \; b_2 \; b_3 \; \cdots$$

Inductively define new enumerations

$$a'_1 \; a'_2 \; a'_3 \; \cdots \text{ and } b'_1 \; b'_2 \; b'_3 \; \cdots$$

such that $a'_i < a'_j$ iff $b'_i < b'_j$. The isomorphism h is $h(a'_i) := b'_i$ for all $i > 0$.
Suppose we defined $a'_1 \cdots a'_n$ and $b'_1 \cdots b'_n$.

Cantor's theorem

Theorem (Cantor)

Any two countable unbounded dense linear orders are isomorphic.

Proof.

Since $\mathcal{A}$ and $\mathcal{B}$ are countable, enumerate their elements:

$$a_1\ a_2\ a_3\ \cdots \text{ and } b_1\ b_2\ b_3\ \cdots$$

Inductively define new enumerations

$$a'_1\ a'_2\ a'_3\ \cdots \text{ and } b'_1\ b'_2\ b'_3\ \cdots$$

such that $a'_i < a'_j$ iff $b'_i < b'_j$. The isomorphism h is $h(a'_i) := b'_i$ for all $i > 0$.

Suppose we defined $a'_1 \cdots a'_n$ and $b'_1 \cdots b'_n$.

If n even, let a'_{n+1} be the first a_i in the enumeration that is different from all $a'_1, \ldots, a'_n$. Define b'_{n+1} such that $a'_i < a'_{n+1}$ iff $b'_i < b'_{n+1}$ for all $1 \leq i \leq n$. Such a b'_{n+1} exists since $\mathcal{B}$ is dense.

Cantor's theorem

Theorem (Cantor)

Any two countable unbounded dense linear orders are isomorphic.

Proof.

Since $\mathcal{A}$ and $\mathcal{B}$ are countable, enumerate their elements:

$$a_1 \; a_2 \; a_3 \; \cdots \text{ and } b_1 \; b_2 \; b_3 \; \cdots$$

Inductively define new enumerations

$$a'_1 \; a'_2 \; a'_3 \; \cdots \text{ and } b'_1 \; b'_2 \; b'_3 \; \cdots$$

such that $a'_i < a'_j$ iff $b'_i < b'_j$. The isomorphism h is $h(a'_i) := b'_i$ for all $i > 0$.
Suppose we defined $a'_1 \cdots a'_n$ and $b'_1 \cdots b'_n$.
If n even, let a'_{n+1} be the first a_i in the enumeration that is different from all $a'_1, \ldots, a'_n$. Define b'_{n+1} such that $a'_i < a'_{n+1}$ iff $b'_i < b'_{n+1}$ for all $1 \leq i \leq n$. Such a b'_{n+1} exists since $\mathcal{B}$ is dense.
If n odd, proceed analogously, starting with b'_{n+1}. Alternation guarantees we process all a_i and b_i. □

Partial Isomorphisms

Definition

Let $\mathcal{A}, \mathcal{B}$ be σ-structures. A partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ is an injective map $h : A \to U_{\mathcal{B}}$ where $A \subseteq U_{\mathcal{A}}$ such that for all P and $a_1, \ldots, a_n \in A$

$$(a_1, \ldots, a_n) \in P_{\mathcal{A}} \iff (h(a_1), \ldots, h(a_n)) \in P_{\mathcal{B}} .$$

Ehrenfeucht-Fraïssé Games

Definition

Given structures $\mathcal{A}, \mathcal{B}$ such that $U_\mathcal{A} \cap U_\mathcal{B} = \emptyset$. The **Ehrenfeucht-Fraïssé game** $G_m(\mathcal{A}, \mathcal{B})$ is played over m **rounds** by two players, Spoiler and Duplicator, according to the following rules:

- In round i, Spoiler chooses element $a_i \in U_\mathcal{A}$ **or** $b_i \in U_\mathcal{B}$
- Then Duplicator answers with an element b_i or a_i from the opposite structure
- After i round, obtain configuration $(a_1, b_1), \ldots, (a_i, b_i)$, giving a remaining game $G_{m-i}(\mathcal{A}, a_1, \ldots, a_i, \mathcal{B}, b_1, \ldots, b_i)$
- Duplicator **wins after** m **rounds** iff $h(a_i) := b_i$ for all $1 \leq i \leq m$ is a partial isomorphism

Winning strategies

Definition

A **winning strategy** for Spoiler maps a configuration $(a_1, b_1), \ldots, (a_i, b_i)$ to an element $a_{i+1} \in U_\mathcal{A}$ or $b_{i+1} \in U_\mathcal{B}$ such that Spoiler is guaranteed to win after m rounds. A winning strategy for Duplicator is defined analogously.

Exercise: Show that either Spoiler or Duplictor has a winning strategy.

Winning strategies

Definition

A **winning strategy** for Spoiler maps a configuration $(a_1, b_1), \ldots, (a_i, b_i)$ to an element $a_{i+1} \in U_{\mathcal{A}}$ or $b_{i+1} \in U_{\mathcal{B}}$ such that Spoiler is guaranteed to win after m rounds. A winning strategy for Duplicator is defined analogously.

Exercise: Show that either Spoiler or Duplictor has a winning strategy.

Example

Let $\mathcal{A} = (\mathbb{Z}, <)$ and $\mathcal{B} = (\mathbb{R}, <)$. Then Duplicator has a winning strategy for $G_2(\mathcal{A}, \mathcal{B})$ but Spoiler has a winning strategy for $G_3(\mathcal{A}, \mathcal{B})$.

Theorem of Ehrenfeucht and Fraïssé

Theorem (Ehrenfeucht, Fraïssé)

For all $m \in \mathbb{N}$, $\mathcal{A} \equiv_m \mathcal{B}$ iff Duplicator wins $G_m(\mathcal{A}, \mathcal{B})$.

Theorem of Ehrenfeucht and Fraïssé

Theorem (Ehrenfeucht, Fraïssé)

For all $m \in \mathbb{N}$, $\mathcal{A} \equiv_m \mathcal{B}$ iff Duplicator wins $G_m(\mathcal{A}, \mathcal{B})$.

Only show that if $\mathcal{A} \not\equiv_m \mathcal{B}$ then Spoiler wins $G_m(\mathcal{A}, \mathcal{B})$:

Proposition

Spoiler has a winning strategy for a game $G_m(\mathcal{A}, a_1, \ldots, a_r, \mathcal{B}, b_1, \ldots, b_r)$ if there is a formula F with free variables $x_1, \ldots, x_r$ and quantifier-depth m such that

$$\mathcal{A}_{[x_1 \mapsto a_1]\cdots[x_r \mapsto a_r]} \models F \text{ and } \mathcal{B}_{[x_1 \mapsto b_1]\cdots[x_r \mapsto b_r]} \models \neg F.$$

An example

Example

Structures $\mathcal{A} = (\mathbb{Z}, <)$ and $\mathcal{B} = (\mathbb{R}, <)$ can be distinguished by a sentence of quantifier depth three:

$$F = \exists x \exists y (x < y \wedge \forall z (\neg (x < z \wedge z < y))).$$

Winning strategy for Spoiler obtained from F:

- Spoiler first chooses $a_1, a_2 \in U_{\mathcal{A}}$ with $a_1 + 1 = a_2$
- Duplicator has to answer with $b_1, b_2 \in U_{\mathcal{B}}$ such that $b_1 < b_2$
- But now Spoiler plays some b_3 such that $b_1 < b_3 < b_2$
- Any a_3 the Duplicator chooses makes her lose

Theorem of Ehrenfeucht and Fraïssé

Proof.

By induction on m. If $m = 0$, then $h(a_i) := b_i$ for all $1 \leq i \leq m$ is not a partial isomorphism.

Theorem of Ehrenfeucht and Fraïssé

Proof.

By induction on m. If $m = 0$, then $h(a_i) := b_i$ for all $1 \leq i \leq m$ is not a partial isomorphism.
Suppose $m > 0$ and $\mathcal{A}_{[x_1 \mapsto a_1] \cdots [x_r \mapsto a_r]} \models F$ and $\mathcal{B}_{[x_1 \mapsto b_1] \cdots [x_r \mapsto b_r]} \models \neg F$.
Then, there is subformula in F of quantifier-depth less than m or of the form $\exists y\, H$ such that H has quantifier depth $m - 1$ that distinguishes $\mathcal{A}$ from $\mathcal{B}$.

Theorem of Ehrenfeucht and Fraïssé

Proof.

By induction on m. If $m = 0$, then $h(a_i) := b_i$ for all $1 \leq i \leq m$ is not a partial isomorphism.
Suppose $m > 0$ and $\mathcal{A}_{[x_1 \mapsto a_1] \cdots [x_r \mapsto a_r]} \models F$ and $\mathcal{B}_{[x_1 \mapsto b_1] \cdots [x_r \mapsto b_r]} \models \neg F$.
Then, there is subformula in F of quantifier-depth less than m or of the form $\exists y\, H$ such that H has quantifier depth $m - 1$ that distinguishes $\mathcal{A}$ from $\mathcal{B}$. In the former case, we obtain a winning strategy for Spoiler by the induction hypothesis.

Theorem of Ehrenfeucht and Fraïssé

Proof.

By induction on m. If $m = 0$, then $h(a_i) := b_i$ for all $1 \leq i \leq m$ is not a partial isomorphism.
Suppose $m > 0$ and $\mathcal{A}_{[x_1 \mapsto a_1]\cdots[x_r \mapsto a_r]} \models F$ and $\mathcal{B}_{[x_1 \mapsto b_1]\cdots[x_r \mapsto b_r]} \models \neg F$.
Then, there is subformula in F of quantifier-depth less than m or of the form $\exists y\, H$ such that H has quantifier depth $m - 1$ that distinguishes $\mathcal{A}$ from $\mathcal{B}$. In the former case, we obtain a winning strategy for Spoiler by the induction hypothesis. Otherwise, we have either

$$(1)\ \mathcal{A} \models \exists y\, H \text{ and } \mathcal{B} \models \forall y \neg H \quad \text{or} \quad (2)\ \mathcal{A} \models \forall y \neg H \text{ and } \mathcal{B} \models \exists y\, H$$

Theorem of Ehrenfeucht and Fraïssé

Proof.

By induction on m. If $m = 0$, then $h(a_i) := b_i$ for all $1 \leq i \leq m$ is not a partial isomorphism.
Suppose $m > 0$ and $\mathcal{A}_{[x_1 \mapsto a_1] \cdots [x_r \mapsto a_r]} \models F$ and $\mathcal{B}_{[x_1 \mapsto b_1] \cdots [x_r \mapsto b_r]} \models \neg F$. Then, there is subformula in F of quantifier-depth less than m or of the form $\exists y\, H$ such that H has quantifier depth $m - 1$ that distinguishes $\mathcal{A}$ from $\mathcal{B}$. In the former case, we obtain a winning strategy for Spoiler by the induction hypothesis. Otherwise, we have either

(1) $\mathcal{A} \models \exists y\, H$ and $\mathcal{B} \models \forall y \neg H$ or (2) $\mathcal{A} \models \forall y \neg H$ and $\mathcal{B} \models \exists y\, H$

In case (1), a winning strategy for Spoiler chooses $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[y \mapsto a][x_1 \mapsto a_1] \cdots [x_r \mapsto a_r]} \models H$, and hence for all $b \in U_{\mathcal{B}}$ we have $\mathcal{A}_{[y \mapsto b][y_1 \mapsto b_1] \cdots [y_r \mapsto b_r]} \models \neg H$. By IH, Spoiler has winning strategy for $G_{m-1}(\mathcal{A}, a_1, \ldots, a_r, a, \mathcal{B}, b_1, \ldots, b_r, b)$, yielding a winning strategy for G_m.

Theorem of Ehrenfeucht and Fraïssé

Proof.

By induction on m. If $m = 0$, then $h(a_i) := b_i$ for all $1 \leq i \leq m$ is not a partial isomorphism.
Suppose $m > 0$ and $\mathcal{A}_{[x_1 \mapsto a_1] \cdots [x_r \mapsto a_r]} \models F$ and $\mathcal{B}_{[x_1 \mapsto b_1] \cdots [x_r \mapsto b_r]} \models \neg F$.
Then, there is subformula in F of quantifier-depth less than m or of the form $\exists y\, H$ such that H has quantifier depth $m - 1$ that distinguishes $\mathcal{A}$ from $\mathcal{B}$. In the former case, we obtain a winning strategy for Spoiler by the induction hypothesis. Otherwise, we have either

$$(1)\ \mathcal{A} \models \exists y\, H \text{ and } \mathcal{B} \models \forall y \neg H \quad \text{or} \quad (2)\ \mathcal{A} \models \forall y \neg H \text{ and } \mathcal{B} \models \exists y\, H$$

In case (1), a winning strategy for Spoiler chooses $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[y \mapsto a][x_1 \mapsto a_1] \cdots [x_r \mapsto a_r]} \models H$, and hence for all $b \in U_{\mathcal{B}}$ we have $\mathcal{A}_{[y \mapsto b][y_1 \mapsto b_1] \cdots [y_r \mapsto b_r]} \models \neg H$. By IH, Spoiler has winning strategy for $G_{m-1}(\mathcal{A}, a_1, \ldots, a_r, a, \mathcal{B}, b_1, \ldots, b_r, b)$, yielding a winning strategy for G_m. In case (2), Spoiler chooses $b \in U_{\mathcal{B}}$ and the proof proceeds symmetrically. □

Applications of Ehrenfeucht and Fraïssé Games

- Concise proof that unbounded dense linear orders are isomorphic (Exercise: establish winning strategy for Duplicator)
- First-order logic cannot express graph connectivity
- The class of all infinite structures is not finitely axiomatisable
- ...and many more