Lecture 14

Decidable Theories

Logical theories, quantifier elimination, unbounded dense linear orders, linear arithmetic over the rationals, Presburger arithmetic

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(with small changes by Javier Esparza)
Logical theories

- Fix a (finite or infinite) signature $\sigma$. We implicitly assume that all formulas are over $\sigma$. We call a closed formula a \textbf{sentence}.

- A \textbf{theory} $\mathcal{T}$ is a set of sentences closed under semantic entailment:
  \[ \mathcal{T} \models F \text{ implies } F \in \mathcal{T} \]

- Given a $\sigma$-structure $\mathcal{A}$, the \textbf{theory of} $\mathcal{A}$, denoted $\text{Th}(\mathcal{A})$, is the theory containing all sentences $F$ such that $\mathcal{A} \models F$.

- The \textbf{theory of a set} $S$ of sentences is the set of sentences $\mathcal{T} = \{ F : S \models F \}$.

- If $\mathcal{T}$ is the theory of $S$, then $S$ is a set of \textbf{axioms} of $\mathcal{T}$. 
Logical theories

A set $\mathcal{F}$ of formulas is **decidable** if there is an algorithm that decides for every formula $F$ whether $F \in \mathcal{F}$ holds.

A theory $\mathcal{T}$ is

- **consistent** if for every sentence $F$, either $F \notin \mathcal{T}$ or $\neg F \notin \mathcal{T}$ (or both).

- **complete** if for every sentence $F$ either $F \in \mathcal{T}$ or $\neg F \in \mathcal{T}$ (or both).

- **decidable** if $\mathcal{T}$ is decidable as a set of sentences.

- **(finitely) axiomatizable** if it is the theory of a (finite) decidable set of sentences.
Logical theories

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- For every $\sigma$-structure $A$, the theory $\text{Th}(A)$ is consistent and complete.
- The only inconsistent theory is the theory of all sentences.
- Theory consistent but not complete: theory of all valid sentences.
- For every theory $\mathcal{T}$, the set $\mathcal{T}$ is a set of axioms of $\mathcal{T}$. 

Logical theories
Examples of logical theories

Example

Linear arithmetic over the rationals is the structure-based theory

\[ \text{Th}(\mathbb{Q}, 1, <, +, \{ c \cdot \}_{c \in \mathbb{Q}}) \]

It allows one to express

- the system of linear inequalities \( Ax \leq b \) has no solution
- every solution of \( Ax \leq b \) is also a solution of \( Cx \leq d \)
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Example

The theory $\mathcal{T}_{\text{UDLO}}$ of unbounded dense linear orders is the set of sentences entailed by the following set of axioms:

$$
F_1 \quad \forall x \forall y \ (x < y \rightarrow \neg (x = y \lor y < x)) \\
F_2 \quad \forall x \forall y \forall z \ (x < y \land y < z \rightarrow x < z) \\
F_3 \quad \forall x \forall y \ (x < y \lor y < x \lor x = y) \\
F_4 \quad \forall x \forall y \ (x < y \rightarrow \exists z \ (x < z \land z < y)) \\
F_5 \quad \forall x \exists y \exists z \ (y < x < z).
$$
Decidable theories

- Quantifier-elimination is a technique to show decidability

- A theory $\mathcal{T}$ admits quantifier-elimination if for any formula (not necessarily a sentence!) $\exists x \, F$ with $F$ quantifier-free, there is a quantifier-free formula $G$ such that

$$\mathcal{T} \models \exists x \, F \iff G$$

- A quantifier-elimination procedure for $\mathcal{T}$ is an algorithm that on input $\exists x \, F$ computes such a formula $G$.

- If $\mathcal{T}$ has
  - a quantifier elimination procedure, and
  - a procedure for deciding $F \in \mathcal{T}$ for variable-free atomic formulas $F$, then $\mathcal{T}$ is decidable.
The theory $T_{UDLO}$ of unbounded dense linear orders is decidable.
Unbounded Dense Linear Orders

Theorem

The theory $\mathcal{T}_{UDLO}$ of unbounded dense linear orders is decidable.

Proof.

Suffices to give quantifier-elimination procedure for $\exists x \, F$, where $F$ is conjunction of atomic formulas $x = y$, $x < y$ or $y < x$ for some variable $y$. After eliminating all quantifiers, end up with Boolean combination of true and false whose truth value can easily be computed.
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Suffices to give quantifier-elimination procedure for $\exists x \ F$, where $F$ is conjunction of atomic formulas $x = y$, $x < y$ or $y < x$ for some variable $y$. Excluding trivial cases, we have

$$F = \bigwedge_{i=1}^{m} l_i < x \land \bigwedge_{j=1}^{n} x < u_j.$$
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If $m = 0$ or $n = 0$ then $\mathcal{T}_{UDLO} \models \exists F \iff \text{true}$. Otherwise

$$\mathcal{T}_{UDLO} \models \exists x F \iff \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{n} l_i < u_j.$$
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Presburger arithmetic

**Figure:** Mojzesz Presburger (1904 - 1943)

\[ \text{Th}(\mathbb{N}, 0, 1, +, <) \] is commonly known as **Presburger arithmetic**.
Simple number theory in Presburger arithmetic

Example

Every natural number is odd or even:

\[ \forall x \exists y (x = y + y \lor x = y + y + 1). \]
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Consider the Chicken McNugget problem: Given \( a_1, \ldots, a_n \in \mathbb{N} \), is there some \( c \in \mathbb{N} \) such that all numbers greater than \( c \) can be represented as a non-negative linear combination of \( a_1, \ldots, a_n \):
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\[ \exists x \forall y (x < y \rightarrow (\exists z_1 \ldots \exists z_n (y = a_1 \cdot z_1 + \cdots + a_n \cdot z_n))). \]
Quantifier-elimination for Presburger arithmetic

- $\text{Th}(\mathbb{N}, 0, 1, +, <)$ does not have quantifier elimination: $y$ cannot be eliminated from $\exists y \ x = y + y$
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- Solution: extend signature with unary divisibility relations $c \mid \cdot$ for all $c > 0$ such that

$$c \mid n \iff \text{there is } k \in \mathbb{N} \text{ such that } n = k \cdot c$$

- $\text{Th}(\mathbb{N}, 0, 1, +, <, \{c \mid \cdot\}_{c > 0})$ has quantifier elimination
Quantifier-elimination for Presburger arithmetic

- Suffices to consider eliminating $x$ from $F = \exists x \land_{1 \leq i \leq n} F_i$
Quantifier-elimination for Presburger arithmetic

- Suffices to consider eliminating $x$ from $F = \exists x \wedge_{1 \leq i \leq n} F_i$

- Rearrange matrix of $F$ so that $x$ is isolated:

$$ F \equiv \exists x \bigwedge_{i \in G} q_i(\vec{y}) < a_i \cdot x \wedge \bigwedge_{j \in L} a_j \cdot x < p_j(\vec{y}) \wedge \bigwedge_{k \in D} c_k | a_k \cdot x + r_k(\vec{y}). $$
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- Let $b = \text{lcm}\{a_i \mid i \in G \cup L \cup D\}$, then $F$ is equi-satisfiable with

$$H = \exists x \land_{i \in G} \frac{b}{a_i} \cdot q_i(\vec{y}) < x \land \land_{j \in L} x < \frac{b}{a_j} \cdot p_j(\vec{y}) \land \land_{k \in D} \frac{b}{a_k} \cdot c_k | x + \frac{b}{a_k} \cdot r_k(\vec{y}) \land b | x.$$
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- Define $c = \text{lcm}\{b, b \cdot c_k/a_k : k \in D\}$, then $H$ is equivalent to

$$\left\{ \begin{array}{ll} \bigvee_{0 \leq m < c} H[m/x] & \text{if } G = \emptyset \\ \bigvee_{j \in G} \bigvee_{1 \leq m \leq c} H[((b/a_j) \cdot q_j(\vec{y}) + m)/x] & \text{otherwise} \end{array} \right.$$
Quantifier-elimination for Presburger arithmetic

Theorem (Oppen)

Presburger arithmetic is decidable in time $2^{2^O(n)}$. 