Lecture 13

Compactness for predicate logic
The compactness theorem, non-standard models of arithmetic

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(with small changes by Javier Esparza)
The compactness theorem

**Theorem**

Let $S$ be a countably infinite set of first-order formulas. Then $S$ is satisfiable if and only if every finite subset of $S$ is satisfiable.
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**Proof.**

Let \( \mathcal{T} \) be obtained from skolemising \( S \), and let \( \mathcal{E} \) be the Herbrand expansion of \( \mathcal{T} \). Then

1. all finite subsets of \( S \) are satisfiable
2. \( \Rightarrow \) all finite subsets of \( \mathcal{T} \) are satisfiable
3. \( \Rightarrow \) all finite subsets of \( \mathcal{E} \) are satisfiable
4. \( \Rightarrow \mathcal{E} \) is satisfiable
5. \( \Rightarrow \mathcal{T} \) is satisfiable
6. \( \Rightarrow S \) is satisfiable.
Justification of individual proof steps

- Not immediate how to construct $\mathcal{T}$, since $S$ could use up all function symbols $f_1, f_2, \ldots$
- Rename $f_i$ to $f_{2i}$ to ensure there are infinitely many unused function symbols $f_{2i+1}, i \geq 0$ available
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- $(1) \Rightarrow (2)$ since skolemisation preserves satisfiability

- $(2) \Rightarrow (3)$ cf. ground resolution

- $(3) \Rightarrow (4)$ since propositional logic is compact

- $(4) \Rightarrow (5)$ a propositional model $\mathcal{A}$ for $\mathcal{E}$ induces a Herbrand model in which in particular

  $$(t_1, \ldots, t_k) \in P_\mathcal{H} \iff \mathcal{A} \models P(t_1, \ldots, t_k)$$

- $(5) \Rightarrow (6)$ analogue to proof that skolemisation preserves satisfiability
Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.
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Example
Let $F$ be a $\sigma$-sentence over some signature $\sigma$ such that $F$ has a model $A_n$ with $|U_{A_n}| = n$ for every $n > 1$. Then $F$ has a model with an infinite universe.
Applications of the compactness theorem

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Proof.

Introduce fresh binary predicate $R$, and for $n > 1$ define

$G_n = \forall x \neg R(x, x) \land \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j)$. In particular,

$B \models G_n$ implies $|U_B| \geq n$. 

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Moreover, $F_n = F \land G_n$ is satisfiable for every $n > 1$. 
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Moreover, $F_n = F \land G_n$ is satisfiable for every $n > 1$. Now define $S := \bigcup_{n>1} \{F_n\}$. Every finite subset of $S$ is satisfiable, hence $S$ has model $\mathcal{B}$. If $|U_\mathcal{B}|$ were equal to some $n \in \mathbb{N}$ then $\mathcal{B} \not\models F_{n+1}$. 
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Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$ be the signature of arithmetic. Can we find a possibly infinite set of $\sigma$-formulas whose only model up to isomorphism is the classical arithmetic?
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**Figure:** Giuseppe Peano (1858 - 1932)
**Peano axioms**

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

\[
\begin{align*}
\forall x \neg (s(x) = 0) \\
\forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
\forall x x + 0 = x \\
\forall x \forall y x + s(y) = s(x + y) \\
\forall x \forall y (s(x) = s(y) \rightarrow x = y) \\
\forall x x \cdot 0 = 0 \\
\forall x \forall y (x \cdot s(y) = (x \cdot y) + x)
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& \quad \forall x \forall y (x \cdot s(y) = (x \cdot y) + x)
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Induction over natural numbers:

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\forall Y (0 \in Y \land \forall x(x \in Y \implies s(x) \in Y)) \implies \forall x x \in Y.
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\forall x x + 0 &= x & \forall x x \cdot 0 &= 0 \\
\forall x x + s(y) &= s(x + y) & \forall x \forall y (x \cdot s(y) &= (x \cdot y) + x)
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Problem: Cannot quantify over sets in first-order logic, instead resort to induction scheme for all formulas \( \phi(x, y_1, \ldots, y_K) \):

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\forall y_1 \ldots y_K (\phi(0) \land \forall x (\phi(x) \rightarrow \phi(s(x)))) \rightarrow \forall x \phi(x).
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Let \( S_{PA} \) be the union of all formulas above, then “classical arithmetic” is a model of \( S_{PA} \).
Theorem (without proof)

*First-order logic with equality is compact.*
Non-standard models of arithmetic

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First-order logic with equality is compact.

To obtain a non-standard model arithmetic:

- Introduce new constant symbol $c$ and set
  \[
  \mathcal{C} = \{ \neg(c = s^i(0)) : i \in \mathbb{N} \}.
  \]

- Every finite subset of $S_{PA} \cup \mathcal{C}$ is satisfiable, hence $S_{PA} \cup \mathcal{C}$ has model $\mathcal{A}$

- But $c_\mathcal{A} \neq s^i_\mathcal{A}(0)$ for all $i \in \mathbb{N}$

- Hence $\mathcal{A}$ is not isomorphic to the standard “classical” model of arithmetic
The Löwenheim-Skolem theorems

**Theorem (Upward Löwenheim-Skolem theorem)**

If $S$ has an infinite model $\mathcal{A}$ then for any cardinal $\kappa$ it has a model $\mathcal{B}$ with a universe of cardinality $\kappa$ that extends $\mathcal{A}$.

**Theorem (Downward Löwenheim-Skolem theorem)**

If $S$ has an infinite model $\mathcal{A}$ then it has a model $\mathcal{B}$ with a countable universe which is a substructure of $\mathcal{A}$.
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**Corollary**

Classical arithmetic is not first-order axiomatisable.