Lecture 13

Compactness for predicate logic

The compactness theorem, non-standard models of arithmetic

Dr Christoph Haase
University of Oxford
(with small changes by Javier Esparza)

The compactness theorem

Theorem

Let $\mathcal{S}$ be a countably infinite set of first-order formulas. Then $\mathcal{S}$ is satisfiable if and only if every finite subset of $\mathcal{S}$ is satisfiable.

The compactness theorem

Theorem

Let $\mathcal{S}$ be a countably infinite set of first-order formulas. Then $\mathcal{S}$ is satisfiable if and only if every finite subset of $\mathcal{S}$ is satisfiable.

Proof.

Let $\mathcal{T}$ be obtained from skolemising $\mathcal{S}$, and let $\mathcal{E}$ be the Herbrand expansion of $\mathcal{T}$. Then

$$
\begin{array}{ll}
(1) & \text{all finite subsets of } \mathcal{S} \text{ are satisfiable} \\
(2) & \Rightarrow \text{all finite subsets of } \mathcal{T} \text{ are satisfiable} \\
(3) & \Rightarrow \text{all finite subsets of } \mathcal{E} \text{ are satisfiable} \\
(4) & \Rightarrow \mathcal{E} \text{ is satisfiable} \\
(5) & \Rightarrow \mathcal{T} \text{ is satisfiable} \\
(6) & \Rightarrow \mathcal{S} \text{ is satisfiable.}
\end{array}
$$

□

Justification of individual proof steps

- Not immediate how to construct $\mathcal{T}$, since $\mathcal{S}$ could use up all function symbols $f_1, f_2, \ldots$
- Rename f_i to f_{2i} to ensure there are infinitely many unused function symbols f_{2i+1}, $i \geq 0$ available

Justification of individual proof steps

- Not immediate how to construct $\mathcal{T}$, since $\mathcal{S}$ could use up all function symbols $f_1, f_2, \ldots$
- Rename f_i to f_{2i} to ensure there are infinitely many unused function symbols f_{2i+1}, $i \geq 0$ available
- (1) $\Rightarrow$ (2) since skolemisation preserves satisfiability
- (2) $\Rightarrow$ (3) cf. ground resolution
- (3) $\Rightarrow$ (4) since propositional logic is compact
- (4) $\Rightarrow$ (5) a propositional model $\mathcal{A}$ for $\mathcal{E}$ induces a Herbrand model in which in particular

$$(t_1, \ldots, t_k) \in P_{\mathcal{H}} \iff \mathcal{A} \models P(t_1, \ldots, t_k)$$

- (5) $\Rightarrow$ (6) analogue to proof that skolemisation preserves satisfiability

Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.

Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.

Example

Let F be a σ-sentence over some signature σ such that F has a model $\mathcal{A}_n$ with $|U_{\mathcal{A}_n}| = n$ for every $n > 1$. Then F has a model with an infinite universe.

Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.

Example

Let F be a σ-sentence over some signature σ such that F has a model $\mathcal{A}_n$ with $|U_{\mathcal{A}_n}| = n$ for every $n > 1$. Then F has a model with an infinite universe.

Proof.

Introduce fresh binary predicate R , and for $n > 1$ define $G_n = \forall x \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j)$. In particular,

$$\mathcal{B} \models G_n \text{ implies } |U_{\mathcal{B}}| \geq n.$$

Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.

Example

Let F be a σ-sentence over some signature σ such that F has a model $\mathcal{A}_n$ with $|U_{\mathcal{A}_n}| = n$ for every $n > 1$. Then F has a model with an infinite universe.

Proof.

Introduce fresh binary predicate R , and for $n > 1$ define $G_n = \forall x \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j)$. In particular,

$$\mathcal{B} \models G_n \text{ implies } |U_{\mathcal{B}}| \geq n.$$

Moreover, $F_n = F \wedge G_n$ is satisfiable for every $n > 1$.

Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.

Example

Let F be a σ-sentence over some signature σ such that F has a model $\mathcal{A}_n$ with $|U_{\mathcal{A}_n}| = n$ for every $n > 1$. Then F has a model with an infinite universe.

Proof.

Introduce fresh binary predicate R , and for $n > 1$ define $G_n = \forall x \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j)$. In particular,

$$\mathcal{B} \models G_n \text{ implies } |U_{\mathcal{B}}| \geq n.$$

Moreover, $F_n = F \wedge G_n$ is satisfiable for every $n > 1$. Now define $\mathcal{S} := \bigcup_{n>1} \{F_n\}$. Every finite subset of $\mathcal{S}$ is satisfiable, hence $\mathcal{S}$ has model $\mathcal{B}$. If $|U_{\mathcal{B}}|$ were equal to some $n \in \mathbb{N}$ then $\mathcal{B} \not\models F_{n+1}$.

Applications of the compactness theorem

Compactness theorem puts limits to the expressive power of first-order formulas.

Example

Let F be a σ-sentence over some signature σ such that F has a model $\mathcal{A}_n$ with $|U_{\mathcal{A}_n}| = n$ for every $n > 1$. Then F has a model with an infinite universe.

Proof.

Introduce fresh binary predicate R , and for $n > 1$ define $G_n = \forall x \neg R(x, x) \wedge \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} R(x_i, x_j)$. In particular,

$$\mathcal{B} \models G_n \text{ implies } |U_{\mathcal{B}}| \geq n.$$

Moreover, $F_n = F \wedge G_n$ is satisfiable for every $n > 1$. Now define $\mathcal{S} := \bigcup_{n>1} \{F_n\}$. Every finite subset of $\mathcal{S}$ is satisfiable, hence $\mathcal{S}$ has model $\mathcal{B}$. If $|U_{\mathcal{B}}|$ were equal to some $n \in \mathbb{N}$ then $\mathcal{B} \not\models F_{n+1}$. Hence $|U_{\mathcal{B}}|$ is infinite, and $\mathcal{B}$ induces model $\mathcal{A}$ of F with an infinite universe. $\square$

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$ be the signature of arithmetic. Can we find a possibly infinite set of σ-formulas whose only model up to isomorphism is the classical arithmetic?

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$ be the signature of arithmetic. Can we find a possibly infinite set of σ-formulas whose only model up to isomorphism is the classical arithmetic?

Figure: Giuseppe Peano (1858 - 1932)

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

$$\forall x\, \neg(s(x) = 0) \qquad \forall x \forall y\, x + s(y) = s(x + y)$$

$$\forall x \forall y\, (s(x) = s(y) \rightarrow x = y) \qquad \forall x\, x \cdot 0 = 0$$

$$\forall x\, x + 0 = x \qquad \forall x \forall y\, (x \cdot s(y) = (x \cdot y) + x)$$

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

$$\forall x \, \neg(s(x) = 0) \qquad \forall x \forall y \; x + s(y) = s(x + y)$$

$$\forall x \forall y \, (s(x) = s(y) \rightarrow x = y) \qquad \forall x \; x \cdot 0 = 0$$

$$\forall x \; x + 0 = x \qquad \forall x \forall y \, (x \cdot s(y) = (x \cdot y) + x)$$

Induction over natural numbers:

$$\forall Y \, (0 \in Y \wedge \forall x (x \in Y \rightarrow s(x) \in Y)) \rightarrow \forall x \; x \in Y.$$

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

$$\forall x\, \neg(s(x) = 0) \qquad \forall x \forall y\, x + s(y) = s(x + y)$$

$$\forall x \forall y\, (s(x) = s(y) \to x = y) \qquad \forall x\, x \cdot 0 = 0$$

$$\forall x\, x + 0 = x \qquad \forall x \forall y\, (x \cdot s(y) = (x \cdot y) + x)$$

Induction over natural numbers:

$$\forall Y\, (0 \in Y \wedge \forall x(x \in Y \to s(x) \in Y)) \to \forall x\, x \in Y.$$

Problem: Cannot quantify over sets in first-order logic, instead resort to induction scheme for all formulas $\phi(x, y_1, \ldots, y_K)$:

$$\forall y_1 \ldots y_k\, (\phi(0) \wedge \forall x(\phi(x) \to \phi(s(x)))) \to \forall x \phi(x).$$

Peano axioms

Let $\sigma = \langle 0, s, +, \cdot, = \rangle$. Peano axioms axiomatise elementary facts about arithmetic in first-order logic with equality:

$$\begin{array}{ll}
\forall x\, \neg(s(x) = 0) & \forall x \forall y\; x + s(y) = s(x + y) \\
\forall x \forall y\, (s(x) = s(y) \to x = y) & \forall x\; x \cdot 0 = 0 \\
\forall x\; x + 0 = x & \forall x \forall y\, (x \cdot s(y) = (x \cdot y) + x)
\end{array}$$

Induction over natural numbers:

$$\forall Y\, (0 \in Y \wedge \forall x(x \in Y \to s(x) \in Y)) \to \forall x\; x \in Y.$$

Problem: Cannot quantify over sets in first-order logic, instead resort to induction scheme for all formulas $\phi(x, y_1, \ldots, y_K)$:

$$\forall y_1 \ldots y_k\, (\phi(0) \wedge \forall x(\phi(x) \to \phi(s(x)))) \to \forall x \phi(x).$$

Let $\mathcal{S}_{PA}$ be the union of all formulas above, then “classical arithmetic” is a model of $\mathcal{S}_{PA}$.

Non-standard models of arithmetic

Theorem (without proof)

First-order logic with equality is compact.

Non-standard models of arithmetic

Theorem (without proof)

First-order logic with equality is compact.

To obtain a non-standard model arithmetic:

- Introduce new constant symbol c and set

$$\mathcal{C} = \{\neg(c = s^i(0)) : i \in \mathbb{N}\}.$$

- Every finite subset of $\mathcal{S}_{PA} \cup \mathcal{C}$ is satisfiable, hence $\mathcal{S}_{PA} \cup \mathcal{C}$ has model $\mathcal{A}$
- But $c_{\mathcal{A}} \neq s^i_{\mathcal{A}}(0_{\mathcal{A}})$ for all $i \in \mathbb{N}$
- Hence $\mathcal{A}$ is not isomorphic to the standard "classical" model of arithmetic

The Löwenheim-Skolem theorems

Theorem (Upward Löwenheim-Skolem theorem)

If $\mathcal{S}$ has an infinite model $\mathcal{A}$ then for any cardinal κ it has a model $\mathcal{B}$ with a universe of cardinality κ that extends $\mathcal{A}$.

Theorem (Downward Löwenheim-Skolem theorem)

If $\mathcal{S}$ has an infinite model $\mathcal{A}$ then it has a model $\mathcal{B}$ with a countable universe which is a substructure of $\mathcal{A}$.

The Löwenheim-Skolem theorems

Theorem (Upward Löwenheim-Skolem theorem)

If $\mathcal{S}$ has an infinite model $\mathcal{A}$ then for any cardinal κ it has a model $\mathcal{B}$ with a universe of cardinality κ that extends $\mathcal{A}$.

Theorem (Downward Löwenheim-Skolem theorem)

If $\mathcal{S}$ has an infinite model $\mathcal{A}$ then it has a model $\mathcal{B}$ with a countable universe which is a substructure of $\mathcal{A}$.

Corollary

Classical arithmetic is not first-order axiomatisable.