Lecture 10

Herbrand's theorem and ground resolution

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Recap

Give a quantifier-free formula F:

- Prenex form: $Q_1 x_1 Q_2 x_2 \cdots Q_n x_n F$
- Skolem form: $\forall x_1 \forall x_2 \cdots \forall x_n F$
- Assuming Axiom of Choice, every first-order formula can be translated into an equi-satisfiable formula in Skolem form

Jaques Herbrand (1908 – 1931)

Ground terms

Definition

Given a signature σ, a **ground term** is a σ-term in which no variable symbol appears.

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Example

Let $\sigma = \langle c, d,\ f, g,\ P, Q \rangle$ with unary f and binary g, the set of ground terms is

$$\{c, d, f(c), f(d), g(c, c), g(c, d), g(d, c), g(d, d), f(f(c)), f(f(d)), \ldots\}$$

Herbrand structures

Definition

Let σ be a signature with at least one constant symbol. A σ-structure $\mathcal{H}$ is a **Herbrand structure** if the following hold:

- The universe $U_{\mathcal{H}}$ is the set of ground terms over σ.
- For every constant symbol c, we have $c_{\mathcal{H}} = c$.
- For every function symbol f, for all ground terms $t_1, \ldots, t_n$: $f_{\mathcal{H}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Observe that every formula F can be interpreted over Herbrand structures, even if no constant symbol appears in F! For example, take the Herbrand structures over the signature containing all function and predicate symbols that appear in F, plus one arbitrary constant symbol.

Herbrand structures

Proposition

For every Herbrand structure $\mathcal{H}$ and ground term t, $\mathcal{H}(t) = t$.

Proof.

Straightforward structural induction on terms. $\square$

So: All Herbrand structures interpret ground terms in the same way!

Lemma (Translation Lemma for Herbrand structures)

For every formula F, ground term t, and Herbrand structure $\mathcal{H}$:

$$\mathcal{H} \models F[t/x] \text{ if and only if } \mathcal{H}_{[x \mapsto t]} \models F.$$

Proof.

The Translation Lemma proved previously gives

$$\mathcal{H} \models F[t/x] \text{ if and only if } \mathcal{H}_{[x \mapsto \mathcal{H}(t)]} \models F\ .$$

Use now that $\mathcal{H}(t) = t$. $\square$

Herbrand's theorem

Theorem

Let $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ *be a closed formula in Skolem form. Then F is satisfiable if and only if F has a Herbrand model.*

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Proof.

It suffices to show: If F has a model, then it also has a Herbrand model.
Assume $\mathcal{A}$ is a model of F. We use $\mathcal{A}$ to define a Herbrand structure $\mathcal{H}$, and show it is a model of F.
Since F is closed, the interpretation of variables is irrelevant. So for every variable x pick an arbitrary ground term t_x and set $x_{\mathcal{H}} = t_x$.
It only remains to fix the interpretation $P_{\mathcal{H}}$ of each predicate symbol P. For all ground terms $t_1, \dots, t_k$ we choose:

$$(t_1, \dots, t_k) \in P_{\mathcal{H}} \text{ iff } \mathcal{A} \models P(t_1, \dots, t_k)$$

Observe that, with this choice, for all ground terms $t_1, \dots, t_k$ and predicate symbols P we have: $\mathcal{A} \models P(t_1, \dots, t_k)$ iff $\mathcal{H} \models P(t_1, \dots, t_k)$.
We claim that $\mathcal{H}$ is a model of F. (Continues) □

Herbrand's theorem

Theorem

Let $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ be a closed formula in Skolem form. Then F is satisfiable if and only if F has a Herbrand model.

Proof.

(Continued) We prove that $\mathcal{H}$ is a model of F by induction on n.

Base: $n = 0$. Then $F = F^*$.

Since $F = F^*$ and closed by hypothesis, F is a Boolean combination of atomic formulas $P(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k$ are ground terms.

So, by the observation in the previous slide, $\mathcal{A} \models F$ iff $\mathcal{H} \models F$.

Since $\mathcal{A} \models F$ by assumption, we get $\mathcal{H} \models F$.

□

Herbrand's theorem

Theorem

Let $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ be a closed formula in Skolem form. Then F is satisfiable if and only if F has a Herbrand model.

Proof.

(Continued) We prove that $\mathcal{H}$ is a model of F by induction on n.

Step: $n > 0$. Let $G = \forall x_2 \ldots \forall x_n F^*$. We have $F = \forall x_1\ G$.

Since $U_{\mathcal{H}}$ is the set of ground terms, proving $\mathcal{H} \models F$ amounts to proving $\mathcal{H}_{[x_1 \mapsto t]} \models G$ for every ground term t.

Let t be an arbitrary ground term. We prove $\mathcal{H}_{[x_1 \mapsto t]} \models G$.

Since $\mathcal{A} \models \forall x_1\ G$, we have $\mathcal{A}_{[x_1 \mapsto a]} \models G$ for every $a \in U_{\mathcal{A}}$. In particular $\mathcal{A}_{[x_1 \mapsto \mathcal{A}(t)]} \models G$.

By the (ordinary) Translation Lemma, $\mathcal{A} \models G[t/x_1]$.

Since $G[t/x_1]$ is closed (t is ground!) and in Skolem form, we can apply the induction hypothesis to $G[t/x_1]$, yielding: $\mathcal{H} \models G[t/x_1]$.

By Translation Lemma for Herbrand structures, $\mathcal{H}_{[x_1 \mapsto t]} \models G$. □

Ground resolution

Definition

The **Herbrand expansion** of a formula $F = \forall x_1 \forall x_2 \ldots \forall x_n F^*$ is the set:

$$E(F) := \{F^*[t_1/x_1][t_2/x_2]\ldots[t_n/x_n] : t_1, \ldots, t_n \text{ ground terms}\}$$

Observe: $E(F)$ is a set of propositional formulas over the set of all variables $P(t_1, \ldots, t_k)$, where P appears in F^*, and $t_1, \ldots, t_n$ are ground terms.

Ground resolution

Theorem

A closed formula $F = \forall x_1 \dots \forall x_n\, F^*$ in Skolem form is satisfiable iff $E(F)$ is satisfiable when considered as a set of propositional formulas.

Ground resolution

Theorem

A closed formula $F = \forall x_1 \ldots \forall x_n\ F^*$ *in Skolem form is satisfiable iff* $E(F)$ *is satisfiable when considered as a set of propositional formulas.*

Proof.

By Herbrand's theorem, F is satisfiable iff it has a Herbrand model.
For every Herbrand structure $\mathcal{H}$ we have:

$$\begin{aligned} \mathcal{H} \models F &\text{ iff } \mathcal{H}_{[x_1 \mapsto t_1] \cdots [x_n \mapsto t_n]} \models F^* \text{ for all ground terms } t_1, \ldots, t_n \\ &\text{ iff } \mathcal{H} \models F^*[t_1/x_1] \ldots [t_n/x_n] \text{ (Translation Lemma)} \\ &\text{ iff } \mathcal{H} \models E(F) \\ &\text{ iff } E(F) \text{ satisfiable as set of prop. formulas} \end{aligned}$$

□

Ground resolution

Theorem

A closed formula $F = \forall x_1 \ldots \forall x_n\ F^*$ in Skolem form is satisfiable iff $E(F)$ is satisfiable when considered as a set of propositional formulas.

Proof.

By Herbrand's theorem, F is satisfiable iff it has a Herbrand model. For every Herbrand structure $\mathcal{H}$ we have:

$$\begin{aligned}\mathcal{H} \models F \text{ iff } & \mathcal{H}_{[x_1 \mapsto t_1]\cdots[x_n \mapsto t_n]} \models F^* \text{ for all ground terms } t_1, \ldots, t_n \\ \text{iff } & \mathcal{H} \models F^*[t_1/x_1] \ldots [t_n/x_n] \text{ (Translation Lemma)} \\ \text{iff } & \mathcal{H} \models E(F) \\ \text{iff } & E(F) \text{ satisfiable as set of prop. formulas}\end{aligned}$$

□

Ground resolution

Theorem

A closed formula F in Skolem form is unsatisfiable if and only if there is a propositional resolution proof of $\Box$ from $E(F)$.

Ground resolution

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Proof.

By the compactness theorem, $E(F)$ is unsatisfiable if and only if some finite subset of $E(F)$ is unsatisfiable if and only if $\square$ can be derived from $E(F)$ using resolution. $\square$