Lecture 1

History of mathematical logic in computer science

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(with small changes by Javier Esparza)

Agenda

1. Literature
2. A historical perspective on logic
3. Contemporary highlights of logic in computer science

Literature

Recommended textbooks:

- *'Logic for Computer Scientists'*, U. Schöning
- *'Mathematical Logic for Computer Science'*, M. Ben-Ari
- *'Logic in computer science: modelling and reasoning about systems'*, M. Huth and M. Ryan
- *'Handbook of Practical Logic and Automated Reasoning'*, J. Harrison

Further literature:

- *'Gödel, Escher, Bach: an Eternal Golden Braid'*, D. Hofstadter
- *'Logicomix: An Epic Search for Truth'*, A. Doxiadis and C. Papadimitriou

Aristotle (384–322 BC)

Syllogism: *"...a discourse in which certain (specific) things having been supposed, something different from the things supposed results of necessity because these things are so."*

$$\frac{\text{All } \textit{beings} \text{ are } \textit{mortal} \qquad \text{All } \textit{humans} \text{ are } \textit{beings}}{\text{All } \textit{humans} \text{ are } \textit{mortal}}$$

$$\frac{\text{All } B \text{ are } M \qquad \text{All } H \text{ are } B}{\text{All } H \text{ are } M}$$

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All *beings* are *mortal*
All *humans* are *beings*

All *humans* are *mortal*

All *B* are *M*
All *H* are *B*

All *H* are *M*

Aristotle only gave a compendium of valid arguments.

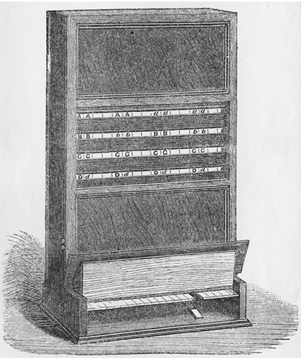
Gottfried Wilhelm Leibniz (1646–1716)

- Envisioned a calculus that allows for testing *any* argument for validity.

"It is obvious that if we could find characters or signs suited for expressing all our thoughts as clearly and as exactly as arithmetic expresses numbers or geometry expresses lines [. . .] all investigations which depend on reasoning would be carried out by transposing these characters and by a species of calculus."

George Boole (1815–1864)

- Equational rules for *propositional logic* in 1854
- Built into the logic piano by William Stanley Jevons

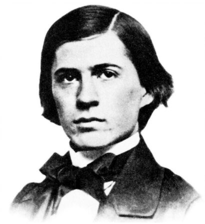


Gottlob Frege (1848–1925) and Charles Sanders Pierce (1839–1914)

- Generalisation of propositional logic to *predicate logic*

"There is just one point where I have encountered a difficulty . . . "

— Bertrand Russell to Frege, 1902



"Beyond doubt . . . he was one of the most original minds of the later nineteenth century, and certainly the greatest American thinker ever."

— Bertrand Russell about Pierce, 1959

Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947)

- Principia Mathematica: attempt to base all of mathematics (set theory, analysis, geometry) rigorously on predicate logic

$*311{\cdot}511.\ \vdash : \text{Infin ax} . \xi \epsilon C`\Theta . Y \epsilon C`H . \supset . (\exists X) . X \epsilon \xi . Y +_g X \sim\epsilon \xi$
[$*311{\cdot}51$. Transp]

$*311{\cdot}52.\ \vdash : \text{Infin ax} . \xi, \eta \epsilon C`\Theta . \supset . \xi \Theta (\xi +_p \eta)$

Dem.

$\vdash . *311{\cdot}511 . \supset \vdash :. \text{Hp} . \supset : Y \epsilon C`H . \supset . (\exists X) . X \epsilon \xi . X +_g Y \sim\epsilon \xi :$
[$*311{\cdot}11$] $\supset : (\exists X, Y) . X +_g Y \epsilon (\xi +_p \eta) - \xi :$
[$*310{\cdot}11 . *311{\cdot}27$] $\supset : \xi \Theta (\xi +_p \eta) :. \supset \vdash .$ Prop

$*311{\cdot}53.\ \vdash : \text{Infin ax} . \xi, \eta \epsilon C`\Theta_n . \supset . \xi \Theta_n (\xi +_p \eta)$ [$*311{\cdot}52{\cdot}33$]

$*311{\cdot}56.\ \vdash :. \text{Infin ax} . \xi \epsilon C`\Theta_g . \supset : \xi = \xi +_p \eta . \equiv . \eta = \iota`0_q$ [$*311{\cdot}1{\cdot}43{\cdot}52{\cdot}53$]

$*311{\cdot}57.\ \vdash :: \text{Infin ax} . \supset :. \xi = \xi +_p \eta . \equiv : \xi = \Lambda . \vee . \xi \epsilon C`\Theta_g . \eta = \iota`0_q$
[$*311{\cdot}56{\cdot}1$]

$*311{\cdot}58.\ \vdash : \text{Infin ax} . \mu \epsilon C`\Theta . \supset . \mu = H``\mu$ [$*304{\cdot}3 . *270{\cdot}31$]

$*311{\cdot}6.\ \vdash : \text{Infin ax} . \mu \Theta \nu . X, Y \epsilon \nu - \mu . XHY . M \epsilon \mu . \supset . M +_g (Y -_s X) \epsilon \nu$

Dem.

$\vdash . *310{\cdot}11 . \supset \vdash : \text{Hp} . \supset . MHX .$
[$*308{\cdot}42{\cdot}72$] $\supset . \{M +_g (Y -_s X)\} HY$ (1)
$\vdash . (1) . *311{\cdot}58 . \supset \vdash .$ Prop

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(proves that 1+1=2 on page 379)

Kurt Gödel (1906–1978)

- No sufficiently strong logical system can be both consistent and complete

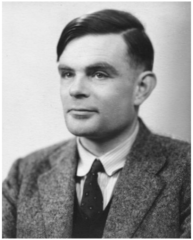
"Kurt Gödel's achievement in modern logic is singular and monumental - indeed it is more than a monument, it is a landmark which will remain visible far in space and time.[. . .] The subject of logic has certainly completely changed its nature and possibilities with Gödel's achievement."

— John von Neumann

Alonso Church (1903–1995) and Alan Turing (1912–1954)

- There is no algorithm that decides whether a given logical argument is valid or not
- *Turing machines* and λ-*calculus* lay the foundations for theoretical computer science.

Alonzo Church, A note on the Entscheidungsproblem", *Journal of Symbolic Logic*, 1 (1936), pp 40-41.



Alan Turing, On computable numbers, with an application to the Entscheidungsproblem, *Proceedings of the London Mathematical Society*, Series 2, 42 (1936-7), pp 230-265.

Claude Shannon (1916–2001) and John Robinson (1930–2016)

Using electrical switches to compute Boolean functions:

Claude Shannon. A Symbolic Analysis of Relay and Switching Circuits, *MSc Thesis*, 1937

Resolution and *unification*: automated reasoning and logic programming:

John Robinson. A Machine-Oriented Logic Based on the Resolution Principle, *Journal of the ACM*, 1965.

Logic is fundamental to computer science

> *When I was a student, even the topologists regarded mathematical logicians as living in outer space. Today the connections between logic and computers are a matter of engineering practice at every level of computer organization.*
>
> *—Martin Davis, "Influences of Mathematical Logic on Computer Science".*

Logic is everywhere

- Hardware design
- Database theory
- Automated verification
- Knowledge representation
- Programming-language theory
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- ...but we won't be focussing on any of the above

This course covers the *foundations* of logic. We will leave applications to subsequent courses, but will emphasize computational aspects: decidability (Models of Computation), and the satisfiability problem (Algorithms)

Finding a needle amongst $1,566 \times 10^{349}$ needles

Erdős discrepancy conjecture

For any $C > 0$ and any infinite sequence $x_1 x_2 x_3 \cdots$ of $+1$'s and -1's there exist $d, k \in \mathbb{N}$ such that

$$|\sum_{1 \leq i \leq k} x_{id}| > C.$$

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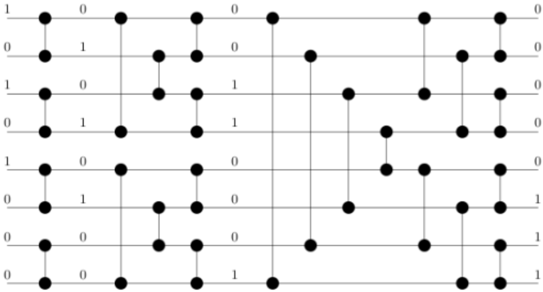
- First shown to hold for $C = 1$ by A.R.D. Mathias in 1993
- Investigated as a Polymath project in 2009-10:

  > *"Given how long a finite sequence can be, it seems unlikely that we could answer this question [for $C = 2$] just by a clever search of all possibilities on a computer."*

- B. Konev and A. Lisitsa solve case $C = 2$ in 2014 using SAT solver
- 13Gb of proof that no sequence of length at least 1161 with discrepancy 2 exists
- Conjecture proved for all $C > 0$ by T. Tao in 2015

Optimal sorting networks



- Consist of wires and comparators, input flows from left to right
- Depth is the maximum number of comparators an input can encounter
- D. Knuth and R. Floyd proved optimality for $n = 1, \ldots, 8$ in 1960s
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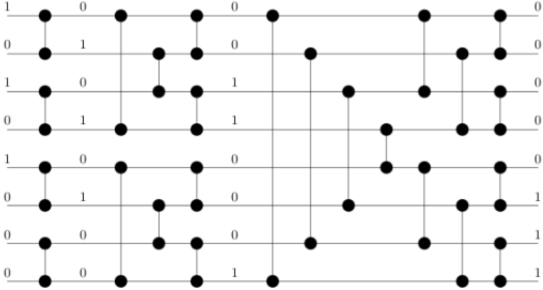

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n	5	6	7	8	9	10	11	12	13	14	15	16
d	5	5	6	6	7	7	8	8	9	9	9	9

Leibniz's ontological proof

Leibniz's proof for the existence of god

1. Definition: God is a being having all perfections.
2. Definition: A perfection is a simple and absolute property.
3. Existence is a perfection.
4. If existence is part of the essence of a thing, then it is a necessary being.
5. If it is possible for a necessary being to exist, then a necessary being does exist.
6. It is possible for a being to have all perfections.
7. Therefore, a necessary being (God) does exist.

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- Leibniz's "algebra of concepts" formalised in the theorem prover Isabelle/HOL by Bentert, Benzmüller, Streit and Woltzenlogel Paleo in 2016
- Showed validity and invalidity of Leibniz's proof in the algebra of concepts depending on the interpretation of some informal statements