Automata and Formal Languages — Retake Exam

• You have 120 minutes to complete the exam.
• Answers must be written in a separate booklet. Do not answer on the exam.
• Please let us know if you need more paper.
• Write your name and Matrikelnummer on every sheet.
• Write with a non-erasable pen. Do not use red or green.
• You are not allowed to use auxiliary means other than pen and paper.
• You can obtain 40 points. You need 17 points to pass.

Question 1 \(( (1 + 1 + 1 + 1) + (2 + 1 + 1 + 1) = 9 \text{ points} )\)

1. Prove: Every regular language can be recognised by an \(\epsilon\)-NFA with a single initial state and a single final state.

2. Disprove: Every \(\omega\)-regular language can be recognised by an NBA with a single initial state and a single accepting state.

3. Prove or disprove: For every regular language \(L\), there exists a planar \(\epsilon\)-NFA \(N\) such that \(L = L(N)\). Recall that a graph is planar if it can be drawn without edge crossings. Note that self-loops are irrelevant for planarity.

4. Let \(\Sigma = \{a, b\}\) be a finite alphabet. Give a sentence of MSO(\(\Sigma\)) for the language \(L = a^*b^*\).

Let \(D\) be a deterministic automaton and \(N\) be a nondeterministic automaton. Let \(L(D)\) denote the language of \(D\) when we interpret \(D\) as a DFA and \(L_\omega(D)\) when we interpret \(D\) as a DBA. Analogously, let \(L(N)\) denote the language of \(N\) when we interpret \(N\) as a NFA and \(L_\omega(N)\) when we interpret \(N\) as a NBA.

5. Prove: \(L(D)\) is infinite if and only if \(L_\omega(D)\) is non-empty.

6. Disprove: \(L(N)\) is infinite if and only if \(L_\omega(N)\) is non-empty.

7. Prove or disprove: The LTL formulas \((Gp) U (Gq)\) and \(G(p U q)\) are equivalent.

8. Give an \(\omega\)-regular expression over the alphabet \(\Sigma = 2^{\{a, b\}}\) for the formula \(FGa \land GFb\).
Question 2  (2 + 3 = 5 points)
The perfect shuffle of two languages $L, L' \in \Sigma^*$ is defined as:

$$L \parallel L' := \{ w = a_1 b_1 \cdots a_n b_n \in \Sigma^* : a_1 \cdots a_n \in L \land b_1 \cdots b_n \in L' \}$$

Notice that $a_1 \cdots a_n$ and $b_1 \cdots b_n$ have to have the same length. For example, if $L_1 = \{\epsilon, a, bb\}$ and $L_2 = \{\epsilon, d, cc, eee\}$ then $L_1 \parallel L_2 = \{\epsilon, ad, bcbe\}$.

1. Give a construction that takes two DFAs $A$ and $B$ as input, and returns a DFA accepting $L(A) \parallel L(B)$.
2. Apply your construction to two copies of the following DFA $D$ and give an interpretation of the states of the resulting DFA:

![DFA Diagram]

Question 3  (4 points)
Use the algorithm $UnivNFA$ to test whether the following NFA is universal.

![NFA Diagram]

Question 4  (3 + 3 = 6 points)
Consider languages over the alphabet $\Sigma = \{0, 1\}$ with fixed length of 3.

1. Let $L \subseteq \Sigma^3$ be the set of all prime numbers in the range $[0, 7]$, i.e. $\{2, 3, 5, 7\}$, represented in binary using the msbf encoding. Construct the corresponding fragment of the master automaton for $L$.

2. Decide whether there exists a language $L \subseteq \Sigma^3$ such that the minimal DFA for this language has at least 11 states. Explain your answer.
Question 5 \((3 + 2 + 2 + 4 + 1 = 12\) points\)

A Büchi automaton is very-weak if the only cycles in the automaton (when the transition relation is viewed as a directed graph) are self-loops.

Consider now the following two very-weak NBAs \(N_1\) and \(N_2\):

\[N_1:\]

\[
\begin{array}{c}
q_0 \\
\downarrow b \\
q_1 \\
\downarrow a \quad b \\
q_2 \\
\downarrow a \\
q_3
\end{array}
\]

\[N_2:\]

\[
\begin{array}{c}
p_0 \\
\downarrow a \quad b \\
p_1 \\
\downarrow a \\
p_2 \\
\downarrow b \\
p_3
\end{array}
\]

1. Show that very-weak NBAs are closed under union and intersection by giving constructions for union and intersection. Argue that these constructions yield very-weak NBAs and the correct languages.

2. Apply the intersection construction you defined in (1) to \(N_1\) and \(N_2\).

3. Construct a very-weak NBA for the LTL formula \(\varphi = (a \cup Xb) \lor Fc\).

4. By LTL\((F,X)\) we denote the fragment of LTL with the restricted syntax of:

\[
\varphi := a \mid \neg a \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid F\varphi \mid X\varphi
\]

Sketch a (recursive) procedure that translates LTL\((F,X)\) to very-weak NBAs.

5. Give an LTL formula that cannot be recognised by very-weak NBAs.

Question 6 \((2 + 2 = 4\) points\)

Let \(L \subseteq \Sigma^*\) be a language of finite words. We define the limit \(\omega\)-language \(\overrightarrow{L}\) of a given language \(L\) as follows:

\[w \in \overrightarrow{L} \iff \text{infinitely many prefixes of } w \text{ are in } L\]

For example, if \(L = b + (ab)^*\) then \(\overrightarrow{L} = (ab)^\omega\).

1. Give an NFA \(N\) such that the limit of its language and the language of \(N\) viewed as a Büchi automaton differ.

2. Prove that for any DFA \(D\), the limit of its language and the language of \(D\) viewed as a Büchi automaton coincide.