Exercise 13.1
Let $\mathcal{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathcal{AP}}$. Give Büchi automata for the $\omega$-languages over $\Sigma$ defined by the following LTL formulas:

(a) $\forall x \mathcal{G}\neg p$
(b) $(\mathcal{G} p) \rightarrow (\mathcal{F} q)$
(c) $p \land \neg (\mathcal{X} p)$
(d) $\mathcal{G}(p \mathcal{U} (p \rightarrow q))$
(e) $\mathcal{F} q \rightarrow (\neg q \mathcal{U} (\neg q \land p))$

Exercise 13.2
Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton such that $Q = P \times [n]$ for some finite set $P$ and $n \geq 1$. Automaton $A$ models a system made of $n$ processes. A state $(p, i) \in Q$ represents the current global state $p$ of the system, and the last process $i$ that was executed.

We define two predicates $\text{exec}_j$ and $\text{enab}_j$ over $Q$ indicating whether process $j$ is respectively executed and enabled. More formally, for every $q = (p, i) \in Q$ and $j \in [n]$, let

$$\text{exec}_j(q) \iff i = j,$$

$$\text{enab}_j(q) \iff (p, i) \rightarrow (p', j) \text{ for some } p' \in P.$$

(a) Give LTL formulas over $Q^\omega$ for the following statements:

(i) All processes are executed infinitely often.
(ii) If a process is enabled infinitely often, then it is executed infinitely often.
(iii) If a process is eventually permanently enabled, then it is executed infinitely often.

(b) The three above properties are known respectively as unconditional, strong and weak fairness. Show the following implications, and show that the reverse implications do not hold:

unconditional fairness $\implies$ strong fairness $\implies$ weak fairness.
Exercise 13.3
Prove or disprove:

(a) \( F(\varphi \lor \psi) \equiv F\varphi \lor F\psi \)
(b) \( F(\varphi \land \psi) \equiv F\varphi \land F\psi \)
(c) \( G(\varphi \lor \psi) \equiv G\varphi \lor G\psi \)
(d) \( G(\varphi \land \psi) \equiv G\varphi \land G\psi \)
(e) \((\varphi \lor \psi) U \rho \equiv (\varphi U \rho) \lor (\psi U \rho)\)
(f) \((\varphi \land \psi) U \rho \equiv (\varphi U \rho) \lor (\psi U \rho)\)

Exercise 13.4
Let \( \text{AP} = \{p, q\} \) and let \( \Sigma = 2^{\text{AP}} \). An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) \( Gp \rightarrow Fp \)
(b) \( G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq) \)
(c) \( FGp \lor FG\neg p \)
(d) \( \neg Fp \rightarrow F\neg Fp \)
(e) \( (Gp \rightarrow Fq) \leftrightarrow (p U (\neg p \lor q)) \)
(f) \( (Gp \lor Gq) \leftrightarrow (\neg p U \neg q) \)
(g) \( G(p \rightarrow Xp) \rightarrow (p \rightarrow Gp) \)
(h) \( (GFp \land GFq) \rightarrow G(p U q) \)
(i) \( G(p U q) \rightarrow (GFp \lor GFq) \)
Solution 13.1

(a)

(b) Note that \((\text{GF}p) \rightarrow (\text{F}q) \equiv \neg(\text{GF}p) \lor (\text{F}q) \equiv (\text{FG}\neg p) \lor (\text{F}q)\). We construct Büchi automata for \(\text{FG}\neg p\) and \(\text{F}q\), and take their union:

(c) Note that \(p \land \neg (\text{XF}p) \equiv p \land \text{XG}\neg p\). We construct a Büchi automaton for \(p \land \text{XG}\neg p\):

(d)

(e)

Solution 13.2

(a) (i) \(\bigwedge_{j \in [n]} \text{GF exec}_j\)

(ii) \(\bigwedge_{j \in [n]} (\text{GF enab}_j \rightarrow \text{GF exec}_j)\)

(iii) \(\bigwedge_{j \in [n]} (\text{FG enab}_j \rightarrow \text{GF exec}_j)\)

(b) • Unconditional fairness implies strong fairness. For the sake of contradiction, suppose unconditional fairness holds for some execution \(\sigma\), but not strong fairness. By assumption, there exists \(j \in [n]\) such
that $\sigma \not\models (\text{GF\ enab}_j \rightarrow \text{GF\ exec}_j)$. Thus,

$\sigma \not\models (\text{GF\ enab}_j \rightarrow \text{GF\ exec}_j)$ $\iff$

$\sigma \models \neg(\text{GF\ enab}_j \rightarrow \text{GF\ exec}_j)$ $\iff$

$\sigma \models \neg(\neg\text{GF\ enab}_j \lor \text{GF\ exec}_j)$ $\iff$

$\sigma \models \text{GF\ enab}_j \land \neg\text{GF\ exec}_j$ $\implies$

$\sigma \models \neg\text{GF\ exec}_j$

which contradicts unconditional fairness.

• Strong fairness implies weak fairness. For the sake of contradiction, suppose strong fairness holds for some execution $\sigma$, but not weak fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\text{FG\ enab}_j \rightarrow \text{GF\ exec}_j)$. Thus,

$\sigma \not\models (\text{FG\ enab}_j \rightarrow \text{GF\ exec}_j)$ $\iff$

$\sigma \models \neg(\text{FG\ enab}_j \rightarrow \text{GF\ exec}_j)$ $\iff$

$\sigma \models \neg(\neg\text{FG\ enab}_j \lor \text{GF\ exec}_j)$ $\iff$

$\sigma \models \text{FG\ enab}_j \land \neg\text{GF\ exec}_j$ $\implies$

$\sigma \models \neg(\text{FG\ enab}_j \rightarrow \text{GF\ exec}_j)$ $\iff$

$\sigma \not\models \text{FG\ enab}_j \rightarrow \text{GF\ exec}_j$

which contradicts strong fairness.

• Strong fairness does not imply unconditional fairness. Execution $(p, 1)(q, 2)^\omega$ of the automaton below satisfies strong fairness, but not unconditional fairness.

• Weak fairness does not imply strong fairness. Execution $((p, 1)(q, 1))^\omega$ of the automaton below satisfies weak fairness, but not strong fairness.

![Diagram](image)

Solution 13.3

(a) True, since:

$\sigma \models F(\varphi \lor \psi) \iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \lor \psi)$

$\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi) \lor (\sigma^k \models \psi)$

$\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi) \lor (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi)$

$\iff \sigma \models F\varphi \lor F\psi.$

(b) False. Let $\sigma = \{p\}{q}\emptyset$. We have $\sigma \models Fp \land Fq$ and $\sigma \not\models F(\varphi \land \psi)$.

(c) False. Let $\sigma = \{p\}{q}\emptyset$. We have $\sigma \models G(p \lor q)$ and $\sigma \not\models Gp \lor Gq.$
(d) True, since:
\[ \sigma \models G(\varphi \land \psi) \iff \forall k \geq 0 \sigma^k \models (\varphi \land \psi) \]
\[ \iff \forall k \geq 0 (\sigma^k \models \varphi) \land (\sigma^k \models \psi) \]
\[ \iff (\forall k \geq 0 \sigma^k \models \varphi) \land (\forall k \geq 0 \sigma^k \models \psi) \]
\[ \iff \sigma \models G\varphi \land G\psi. \]

(e) False. Let \( \sigma = \{p\} \{q\} \emptyset^\omega \). We have \( \sigma \models (p \lor q) \ U \ r \) and \( \sigma \not\models (p \ U \ r) \lor (q \ U \ r) \).

(f) True, since:
\[ \sigma \models p \ U \ (\varphi \lor \psi) \iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \lor \psi) \land \forall 0 \leq i < k \sigma^i \models p \]
\[ \iff \exists k \geq 0 \text{ s.t. } ((\sigma^k \models \varphi) \lor (\sigma^k \models \psi)) \land \forall 0 \leq i < k \sigma^i \models p \]
\[ \iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi \land \forall 0 \leq i < k \sigma^i \models p) \lor (\sigma^k \models \psi \land \forall 0 \leq i < k \sigma^i \models p) \]
\[ \iff \sigma \models (p \ U \ \varphi) \lor (p \ U \ \psi). \]

Solution 13.4

(a) \( Gp \rightarrow Fp \) is a tautology since
\[ \sigma \models Gp \iff \forall k \geq 0 \sigma^k \models p \]
\[ \iff \exists k \geq 0 \sigma^k \models p \]
\[ \iff \sigma \models Fp. \]

(b) \( G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq) \) is a tautology. For the sake of contradiction, suppose this is not the case. There exists \( \sigma \) such that
\[ \sigma \models G(p \rightarrow q), \text{ and } \]
\[ \sigma \not\models (Gp \rightarrow Gq). \]  

By (2), we have
\[ \sigma \models Gp, \text{ and } \]
\[ \sigma \not\models Gq. \]

Therefore, there exists \( k \geq 0 \) such that \( p \in \sigma(k) \) and \( q \notin \sigma(k) \) which contradicts (1).

(c) \( FGp \lor FG\neg p \) is not a tautology since it is not satisfied by \( (\{p\}\{q\})^\omega \).

(d) \( \neg Fp \rightarrow F\neg Fp \) is a tautology since \( \varphi \rightarrow F\varphi \) is a tautology for every formula \( \varphi \).

(e) \( (Gp \rightarrow Fq) \equiv (p \ U \ (\neg p \lor q)) \) is a tautology. We have
\[ Gp \rightarrow Fq \equiv \neg Gp \lor Fq \]  
(by def. of implication)
\[ \equiv F\neg p \lor Fq \]
\[ \equiv F(\neg p \lor q) \]
\[ \equiv F(p \rightarrow q) \]  
(by def. of implication)

Therefore, we have to show that
\[ F(p \rightarrow q) \equiv (p \ U \ (p \rightarrow q)). \]

\( \leftarrow \) Let \( \sigma \) be such that \( \sigma \models (p \ U \ (p \rightarrow q)) \). In particular, there exists \( k \geq 0 \) such that \( \sigma^k \models (p \rightarrow q) \). Therefore, \( \sigma \models F(p \rightarrow q) \).

\( \rightarrow \) Let \( \sigma \) be such that \( \sigma \models F(p \rightarrow q) \). Let \( k \geq 0 \) be the smallest position such that \( \sigma^k \models (p \rightarrow q) \). For every \( 0 \leq i < k \), we have \( \sigma^i \models (p \rightarrow q) \) which is equivalent to \( \sigma^i \models p \land \neg q \). Therefore, for every \( 0 \leq i < k \), we have \( \sigma^i \models p \). This implies that \( \sigma \models p \ U \ (p \rightarrow q) \).
(f) \(\neg(p \lor q) \iff \neg(p \land \neg q)\) is not a tautology. Let \(\sigma = \{p\}\{q\}\). We have \(\sigma \models \neg(p \lor q)\) and \(\sigma \models (p \land \neg q)\).

(g) \(G(p \to Xp) \to (p \to Gp)\) is a tautology since
\[
G(p \to Xp) \to (p \to Gp) \equiv \neg G(p \lor Xp) \lor (p \lor Gp) \quad \text{(by def. of implication)}
\equiv \neg F(p \land \neg Xp) \lor (p \lor Gp) \quad \text{(by def. of implication)}
\equiv \neg Fp \to (p \lor (F(p \land X \neg p)))
\equiv \neg Fp \to (p \lor (F(p \land X \neg p)))
\equiv F \neg p \to F \neg p.
\]

(h) \((GFp \land GFq) \Rightarrow G(p \lor q)\) is not a tautology. Here are two counterexamples: \((\{p\} \emptyset \{q\}\) and \(\emptyset \{p, q\}\).

(i) \((p \lor q) \Rightarrow (GFp \lor GFq)\) is a tautology. We prove this by contradiction.
Suppose the formula is not a tautology. Then there exists an execution \(\sigma\) that does not satisfy it, that is, the following holds:
\[
\sigma \not\models G(p \lor q) \Rightarrow (GFp \lor GFq).
\]
Therefore, we have the following:
\[
\sigma \models G(p \lor q), \quad \sigma \not\models GFp \lor GFq. \quad (3)
\]
First, note that from (3) we know the following:
\[
\sigma^k \models p \lor q, \quad \text{for every } k \geq 0. \quad (5)
\]
Second, note that (4) is equivalent to \(\sigma \models F G \neg p \land F G \neg q\), that is \(\sigma \models F G \neg p\) and \(\sigma \models F G \neg q\). Since we have that \(\sigma \models F G \neg q\), by definition of the operator \(F\) we know the following:
\[
\sigma^i \models G \neg q, \quad \text{for some } i \geq 0. \quad (6)
\]
By definition of \(G\) this means the following:
\[
\sigma^j \models \neg q, \quad \text{for every } j \geq i. \quad (7)
\]
Let us now focus again on (5) and on the index \(i\) defined in (6). From (5) we know that also for this particular index \(i\) it holds that \(\sigma^i \models p \lor q\). Therefore, by definition of \(\lor\), we know that there exists an index \(l \geq i\) with \(\sigma^l \models q\). This contradicts (7), and hence our assumption that the formula is not a tautology is wrong. This shows that the formula is indeed a tautology.