Automata and Formal Languages — Exercise Sheet 12

Exercise 12.1
Let $B$ be the following Büchi automaton.

(a) For every state of $B$, give the discovery time and finishing time assigned by a DFS on $B$ starting in $s_0$ (i.e. the moment they first become grey and the moment they become black). Visit successors $s_i$ of a given state in the ascending order of their indices $i$. For example, when visiting the successors of $s_2$, first visit $s_3$ and later $s_4$.

(b) The language of $B$ is not empty. Give the witness lasso found by applying NestedDFS to $B$ following the same convention for the order of successors as above.

(c) Is the execution in (b) optimal? Does there exist an optimal execution of NestedDFS on $B$ with a different order for visiting successors?

Exercise 12.2
Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give LTL formulas for the following $\omega$-languages:

(a) $\Sigma \{p, q\} \{q\}^\omega$

(b) $\Sigma^* \{q\}^\omega$

(c) $\Sigma^* \emptyset \Sigma^\omega$

(d) $\{p\}^* \{p, q\} (\{p\} + \{p, q\})^\omega$

(e) $\{p\}^* \{q\}^* \emptyset^\omega$

Exercise 12.3
Let $AP = \{p, q, r\}$. Give formulas for the computations satisfying the following properties:

(a) if $q$ eventually holds, then $p$ must not hold before $q$ first does.

(b) if $q$ eventually holds, then $p$ holds at some point before $q$ first holds.

(c) $p$ always holds everywhere between $q$ and $r$.

(d) $p$, and only $p$, holds at even positions and $q$, and only $q$, holds at odd positions.
Exercise 12.4
The *weak until* operator $W$ has the following semantics:

- $\sigma \models \phi_1 W \phi_2$ iff there exists $k \geq 0$ such that $\sigma^k \models \phi_2$ and $\sigma^i \models \phi_1$ for all $0 \leq i < k$, or $\sigma^k \models \phi_1$ for every $k \geq 0$.

Prove: $p W q \equiv G(p \lor (p U q)) \equiv F \neg p \rightarrow (p U q) \equiv p U (q \lor Gp)$.
Solution 12.1
a. We note "state[discovery time/finishing time]."


b. The lasso found by NestedDFS from \( s_0 \) is \( s_0s_1s_2s_3s_4s_5s_5 \).

c. Given a non-empty NBA, we use the following definition of optimal execution of NestedDFS: the algorithm reports NONEMPTY at the earliest time such that all the states of a witness lasso have been explored.

The execution given in (b) is non optimal since it does not return the lasso \( s_0s_1s_2s_3s_1 \) which already appeared in the explored subgraph.

There is no execution of NestedDFS which blackens \( s_2 \) before \( s_5 \). But there is an execution of NestedDFS on \( B \) which returns the lasso \( s_0s_1s_2s_3s_5s_5s_5 \) before it has visited the only other witness lasso \( s_0s_1s_2s_3s_1 \) and thus is optimal: the execution which does dfs1 via \( s_0s_1s_2s_4s_5 \), blackens \( s_5 \) then launches dfs2 from \( s_5 \) and finds a cycle. Node \( s_3 \) is not part of the explored subgraph so the algorithm reports NONEMPTY at the earliest time such that all the states of a witness lasso have been explored.

Solution 12.2
(a) \( X(p \land q) \land XXG(\neg p \land q) \)
(b) \( FG(\neg p \land q) \)
(c) \( F(\neg p \land \neg q) \)
(d) \( Gp \land Fq \)
(e) \( (p \land \neg q) U ((\neg p \land q) U G(\neg p \land \neg q)) \)

Solution 12.3
(a) \( Fq \rightarrow (\neg p U q) \)
(b) \( Fq \rightarrow (\neg q U (\neg q \land p)) \)
(c) \( G((q \land XFr) \rightarrow X(p U r)) \)
(d) \( G(\neg r) \land G(p \leftrightarrow \neg q) \land p \land G(p \rightarrow Xq) \land G(q \rightarrow Xp) \)

Solution 12.4
- \( p \models W q \equiv Gp \lor (p U q) \).
  Follows immediately from the definitions.

- \( Gp \lor (p U q) \equiv F \neg p \rightarrow (p U q) \).
  We have: \( Gp \lor (p U q) \equiv \neg (F \neg p) \lor (p U q) \equiv F \neg p \rightarrow (p U q) \).

- \( Gp \lor (p U q) \equiv p U (q \lor Gp) \).
  Assume \( \sigma \models Gp \lor (p U q) \) holds. If \( \sigma \models Gp \), then \( \sigma \models (p U q) (\psi \lor Gp) \) for every \( \varphi, \psi \). If \( \sigma \models p U q \), then \( \sigma \models Gp \lor (p U q) \).

- \( Gp \lor (p U q) \equiv p U (q \lor Gp) \).
  Assume \( \sigma \models p U (q \lor Gp) \). Then there exists \( k \geq 0 \) such that \( \sigma^k \models q \lor Gp \) and \( \sigma^i \models p \) for all \( 0 \leq i < k \).
  If \( \sigma^k \models q \), then \( \sigma \models q \lor Gp \). If \( \sigma^k \models Gp \), then \( \sigma^i \models p \) for all \( 0 \leq i, \) and so \( \sigma \models Gp \).
  Or simply, using the fact given in the lecture that \( \varphi U (\psi_1 \lor \psi_2) \equiv (\varphi U \psi_1) \lor (\varphi U \psi_2) \), we have:
  \( p U (q \lor Gp) \equiv (p U q) \lor (p U Gp) \equiv (p U q) \lor Gp \equiv Gp \lor (p U q) \).