Exercise 10.1
Give deterministic Rabin automata and Muller automata for the following language:

\[ L = \{ w \in \{a, b\}^* : w \text{ contains finitely many } a's \}. \]

Exercise 10.2
Give a procedure that translates non-deterministic Rabin automata to non-deterministic Büchi automata.

Exercise 10.3
Consider the following automaton \( A \):

(a) Interpret \( A \) as a Rabin automaton with acceptance condition \{\{q_0, q_2\}, \{q_1\}\}. Follow the approach from Exercise 10.2 to construct a Büchi automaton that recognizes the same language as \( A \).

(b) Interpret \( A \) as a Muller automaton with acceptance condition \{\{q_1\}, \{q_0, q_2\}\}. Use algorithms NMAtoNGA and NGAtoNBA from the lecture notes to construct a Büchi automaton that recognizes the same language as \( A \).

Exercise 10.4
Consider the class of non deterministic automata over infinite words with the following acceptance condition: an infinite run is accepting if it visits a final state at least once. Show that no such automaton accepts the language of all words over \{a, b\} containing infinitely many a and infinitely many b.
Solution 10.1

- We give the following Rabin automaton with acceptance condition \(\{(q_1, \emptyset), \ldots, (q_n, \emptyset)\}\), i.e. where \(q_1\) must be visited infinitely often and \(q_0\) must be visited finitely often:

- We give the following Muller automaton with acceptance condition \(\{\{q_1\}\}\), i.e. where precisely \(\{q_1\}\) must be visited infinitely often:

Solution 10.2

NBA can be easily transformed into nondeterministic Rabin automata (NRA) and vice versa, without any exponential blow-up.

**NBA \(\rightarrow\) NRA.** Just observe that a Büchi condition \(\{q_1, \ldots, q_k\}\) is equivalent to the following Rabin condition \(\{(q_1, \emptyset), \ldots, (q_n, \emptyset)\}\).

**NRA \(\rightarrow\) NBA.** Given a Rabin automaton \(A = (Q, \Sigma, Q_0, \delta, \{(F_0, G_0), \ldots, (F_m, G_m)\})\), it follows easily that, as in the case of Muller automata, \(L_\omega(A) = \bigcup_{i=0}^{m-1} L_\omega(A_i)\) holds for the NRAs \(A_i = (Q, \Sigma, Q_0, \delta, \{(F_i, G_i)\})\). So it suffices to translate each \(A_i\) into an NBA and take the union of the obtained NBAs. Since an accepting run \(\rho\) of \(A_i\) satisfies \(\inf(\rho) \cap G_i = \emptyset\), from some point on \(\rho\) only visits states of \(Q \setminus G_i\). So \(\rho\) consists of an initial finite part, say \(\rho_0\), that may visit all states, and an infinite part, say \(\rho_1\), that only visits states of \(Q \setminus G_i\). So we take two copies of \(A_i\). Intuitively, \(A'_i\) simulates \(\rho\) by executing \(\rho_0\) in the first copy, and \(\rho_1\) in the second. The condition that \(\rho_1\) must visit some state of \(F_i\) infinitely often is enforced by taking \(F_i\) as Büchi condition.

Solution 10.3

(a)
(b) We must first construct two generalized Büchi automata $A$ and $B$ for $\{q_1\}$ and $\{q_0, q_2\}$ respectively. Automaton $A$ is as follows with acceptance condition $\{\{q_1\}\}$:

![Automaton A](image)

Automaton $B$ is as follows with acceptance condition $\{\{q_0\}, \{q_2\}\}$:

![Automaton B](image)

The resulting generalized Büchi automaton is the union of $A$ and $B$. Note that $A$ is essentially already a standard Büchi automaton, it suffices to make state $[q_1, 1]$ accepting. However, it remains to convert $B$ into a standard Büchi automaton $B'$:

![Automaton B'](image)
Altogether, we obtain the following Büchi automaton:

![Büchi automaton diagram]

★ Since Büchi automata can have multiple initial states, we can also simply take the disjoint union of both automata, i.e. have them side by side instead of adding a single new initial.

**Solution 10.4**

Suppose there is such an automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ recognizing $L$. Since $w = (ab^{|Q|})^\omega$ belongs to $L$, there exist $u, v \in \{a, b\}^*$, $q_{\text{init}} \in Q_0$, $q_{\text{acc}} \in F$, and $q_0, q_1, \ldots, q_{|Q|} \in Q$ such that $uv = (ab^{|Q|})^ma$ for some $m \geq 1$, and

$$q_{\text{init}} \xrightarrow{u} q_{\text{acc}} \xrightarrow{v} q_0 \xrightarrow{b} q_1 \xrightarrow{b} \cdots \xrightarrow{b} q_{|Q|}$$

By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Therefore,

$$q_{\text{init}} \xrightarrow{u} q_{\text{acc}} \xrightarrow{vb^i} q_i \xrightarrow{b^{i-i}} q_j \xrightarrow{b^{i-i}} q_j \xrightarrow{b^{i-i}} \cdots$$

We conclude that $uvb^i(b^{i-i})^\omega$ is accepted by $B$, which is a contradiction.