Automata and Formal Languages — Exercise Sheet 9

Exercise 9.1
Find $\omega$-regular expressions (the shorter the better) for the following languages:

1. $\{w \in \{a,b\}^\omega \mid k \text{ is even for each subword } ba^k b \text{ of } w\}$
2. $\{w \in \{a,b\}^\omega \mid w \text{ has no occurrence of } bab\}$

Exercise 9.2
Give deterministic Büchi automata recognizing the following $\omega$-languages over $\Sigma = \{a,b,c\}$:

- (a) $L_1 = \{w \in \Sigma^\omega : w \text{ contains at least one } c\}$
- (b) $L_2 = \{w \in \Sigma^\omega : \text{in } w, \text{ every } a \text{ is immediately followed by a } b\}$
- (c) $L_3 = \{w \in \Sigma^\omega : \text{in } w, \text{ between two successive } a\text{'s there are at least two } b\text{'s}\}$

Exercise 9.3
Let $\text{inf}(w)$ denote the set of letters occurring infinitely often in the infinite word $w$. Give Büchi automata and $\omega$-regular expressions for the following $\omega$-languages over $\Sigma = \{a,b,c\}$:

- (a) $L_1 = \{w \in \Sigma^\omega : \text{inf}(w) \subseteq \{a,b\}\}$
- (b) $L_2 = \{w \in \Sigma^\omega : \text{inf}(w) = \{a,b\}\}$
- (c) $L_3 = \{w \in \Sigma^\omega : \{a,b\} \subseteq \text{inf}(w)\}$
- (d) $L_4 = \{w \in \Sigma^\omega : \text{inf}(w) = \{a,b,c\}\}$
- (e) $\star$ Does there exist a deterministic Büchi automaton accepting $L_1$? If there is then give it, otherwise give a proof of why it is not true.

Exercise 9.4
Prove or disprove:

- (a) For every Büchi automaton $A$, there exists a Büchi automaton $B$ with a single initial state and such that $L_\omega(A) = L_\omega(B)$;
- (b) For every Büchi automaton $A$, there exists a Büchi automaton $B$ with a single accepting state and such that $L_\omega(A) = L_\omega(B)$;
- (c) There exists a Büchi automaton recognizing the finite $\omega$-language $\{w\}$ such that $w \in \{0,1,\ldots,9\}^\omega$ and $w_i$ is the $i^{th}$ decimal of $\sqrt{2}$. 

Solution 9.1

(1) \(a^*(b^*(aa^*))^\omega\).

(2) \(a^*(b^*(\epsilon + aaa^*))^\omega\) or, one character shorter, \(a^*(b^*(aaa^*))^\omega\).

Solution 9.2

(a)

(b)

(c)

or simply,

Solution 9.3

(a) \((a + b + c)^*(a + b)^\omega\), and

(b) \((a + b + c)^*(aa^*bb^*)^\omega\), and
(c) \((b + c)^*(a + c)^*b)^\omega\), and

\[
\begin{array}{c}
\text{a, b, c} \\
\text{a, b} \\
\text{a, b} \\
\text{c}
\end{array}
\]

or

\[
\begin{array}{c}
\text{a, b} \\
\text{a} \\
\text{a, c} \\
\text{b, c}
\end{array}
\]

(d) \((b + c)^*a(a + c)^*b(a + b)^*c)^\omega\), and

\[
\begin{array}{c}
\text{a, b, c} \\
\text{b, c} \\
\text{a, b} \\
\text{b, c}
\end{array}
\]

or

\[
\begin{array}{c}
\text{a, b, c} \\
\text{c} \\
\text{b} \\
\text{a, c}
\end{array}
\]

(e) \(\star\) It is asked whether there exists a deterministic B"uchi automaton accepting \(L_1\). We show that it is not the case. For the sake of contradiction, suppose there exists a deterministic B"uchi automaton \(B = (Q, \Sigma, \delta, q_0, F)\) such that \(L_\omega(B) = L_1\). Since \(cb^\omega \in L_1\), \(B\) must visit \(F\) infinitely often when reading \(cb^\omega\). In particular, this implies the existence of \(m_1 > 0\) and \(q_1 \in F\) such that \(q_0 \xrightarrow{cb^{m_1}} q_1\). Similarly, since \(cb^{m_1}cb^{m_2} \in L_1\), there exist \(m_2 > 0\) and \(q_2 \in F\) such that \(q_0 \xrightarrow{cb^{m_1}cb^{m_2}} q_2\). Since \(B\) is deterministic, we have \(q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2\). By repeating this argument \(|Q|\) times, we can construct \(m_1, m_2, \ldots, m_{|Q|} > 0\) and \(q_1, q_2, \ldots, q_{|Q|} \in F\) such that

\[
q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2 \cdots \xrightarrow{cb^{m_{|Q|}}} q_{|Q|}.
\]
By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Let
\[ u = c b^{m_1} c b^{m_2} \cdots c b^{m_i}, \]
\[ v = c b^{m_{i+1}} c b^{m_{i+2}} \cdots c b^{m_j}. \]
We have $q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} \cdots$ which implies that $uv^\omega \in L_\omega(B)$. Also notice that $c$ appears infinitely often in $uv^\omega$, that is, $c \in \text{inf}(uv^\omega)$. Therefore we have $uv^\omega \notin L_1 = L_\omega(B)$, which yields a contradiction.

**Solution 9.4**

(a) True. The construction for NFAs still work for Büchi automata.

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton. We add a state to $Q$ which acts as the single initial state. More formally, we define $B' = (Q \cup \{q_{\text{init}}\}, \Sigma, \delta', \{q_{\text{init}}\}, F)$ where
\[ \delta'(q, a) = \begin{cases} \bigcup_{q_0 \in Q_0} \delta(q_0, a) & \text{if } q = q_{\text{init}}, \\ \delta(q, a) & \text{otherwise.} \end{cases} \]
We have $L_\omega(B) = L_\omega(B')$, since there exists $q_0 \in Q_0$ such that
\[ q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots \]
if and only if
\[ q_{\text{init}} \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \xrightarrow{a_3} \cdots. \]

(b) False. Let $L = \{a^\omega, b^\omega\}$. Suppose there exists a Büchi automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ such that $L_\omega(B) = L$ and $F = \{q\}$. Since $a^\omega \in L$, there exist $q_0 \in Q_0$, $m \geq 0$ and $n > 0$ such that
\[ q_0 \xrightarrow{a} q \xrightarrow{a} q. \]
Similarly, since $b^\omega \in L$, there exist $q'_0 \in Q_0$, $m' \geq 0$ and $n' > 0$ such that
\[ q'_0 \xrightarrow{b} q \xrightarrow{b} q. \]
This implies that
\[ q_0 \xrightarrow{a^m} q \xrightarrow{b^n} q. \]
Therefore, $a^m (b^n)^\omega \in L$, which is a contradiction.

(c) False. Suppose there exists a Büchi automaton $B = (Q, \{0, 1, \ldots, 9\}, \delta, Q_0, F)$ such that $L_\omega(B) = \{w\}$. There exist $u \in \{0, 1, \ldots, 9\}^*$, $v \in \{0, 1, \ldots, 9\}^+$, $q_0 \in Q_0$ and $q \in F$ such that
\[ q_0 \xrightarrow{u} q \xrightarrow{v} q. \]
Therefore, $uv^\omega \in L_\omega(B)$ which implies that $w = uv^\omega$. Since $w$ represents the decimals of $\sqrt{2}$, we conclude that $\sqrt{2}$ is rational, which is a contradiction.