Automata and Formal Languages — Endterm Exam

- You have 120 minutes to complete the exam.
- Answers must be written in a separate booklet. Do not answer on the exam.
- Please let us know if you need more paper.
- Write your name and Matrikelnummer on every sheet.
- Write with a non-erasable pen. Do not use red or green.
- You are not allowed to use auxiliary means other than pen and paper.
- You can obtain 40 points. You need 17 points to pass.
- The ★ symbol indicates a more challenging question.

Question 1  (11 points)

a. Let Σ be the alphabet \{a, b\}, and let \(p\) be the word pattern \(ababaa\). Build the DFA \(B_p\) (obtained by determinizing the naive NFA \(A_p\) for \(\Sigma^*p\)).

b. Give the fragment of the master automaton that contains the states of the language \(L = \{aab, bbb, bab\}\) and all its residuals (all the states between \(q_L\) and \(q_\emptyset\)).

c. Given a word \(w\) over the alphabet \(\Sigma = \{a, b\}\) we define \(\overline{w}\) to be the word obtained from \(w\) by replacing \(a\) by \(b\), and \(b\) by \(a\). For example, \(\overline{aababaab} = bbabaaab\) and \(\overline{bab} = abaa\). Decide whether the language \(L = \{w\overline{w}: w \in \Sigma^*\}\) is regular or irregular, and prove this by analyzing its residuals.

d. Give a regular expression recognizing the language of the following the MSO formula

\[ \varphi = \exists x \exists y. x \neq y \land Q_a(x) \land Q_a(y) \land [\forall z. (z \neq x \land z \neq y) \rightarrow (Q_b(z) \land x < z \land z < y)]. \]

e. Consider the following NBA.

![NBA Diagram]

Draw \(\text{dag}(b(baa)^\omega)\). Does it admit an odd ranking? Give such a ranking if it exists, and provide a short justification if it does not.
Question 2 (3 points)
Let $A$ be the following DFA:

(a) Compute the language partitions of $A$.
(b) Draw the minimal automaton using the language partitions from (a).

Question 3 (6 points)
Given $n \in \mathbb{N}$, let $msbf(n)$ be the set of most significant bit first encodings of $n$, i.e., the words that start with an arbitrary number of leading zeros, followed by $n$ written in binary. For example, $msbf(6) = 0^*110$ and $msbf(3) = 0^*11$. Let $val : \{0, 1\}^* \to \mathbb{N}$ be the function that associates to every word $w \in \{0, 1\}^*$ the number $val(w)$ represented by $w$ in the most significant bit first encoding. For example, $val(110) = 6$ and $val(011) = 3$.

a. Let $T$ be the following transducer over alphabet $\Sigma = \{0, 1\} \times \{0, 1\}$.

What is the relation between $val(x)$ and $val(y)$, for any $[x, y]$ accepted by $T$?

b. Draw a transducer $T_{+1}$ recognizing the language

$$\{[x, y] \in \Sigma^* \mid val(y) = val(x) + 1\}.$$
Question 4  (5 points)
Recall: A process can send a message \( m \) to the channel with the instruction \( c!m \). A process can also consume the first message of the channel with the instruction \( c?m \). If the channel is full when executing \( c!m \), then the process blocks and waits until it can send \( m \). When a process executes \( c?m \), it blocks and waits until the first message of the channel becomes \( m \).

Suppose there are two processes being executed concurrently that communicate through a channel \( c \). Channel \( c \) is a queue that can store up to 1 message. The two processes follow these two algorithms respectively:

\[
\begin{align*}
\text{process(1):} & \\
& \text{while } true \text{ do} \\
& \quad c!m \\
& \quad /* \text{ critical section */} \\
& \quad c?m
\end{align*}
\]

\[
\begin{align*}
\text{process(2):} & \\
& \text{while } true \text{ do} \\
& \quad c?m \\
& \quad c!m \\
& \quad /* \text{ critical section */} \\
& \quad c?m
\end{align*}
\]

a. Model the program by constructing a network of three automata:

- One for process 1, using the alphabet \( \Sigma_1 = \{ c?m, c!m, cs_1 \} \),
- One for process 2, using the alphabet \( \Sigma_2 = \{ c?m, c!m, cs_2 \} \),
- One for the channel \( c \) of size 1, that is initially empty, using the alphabet \( \Sigma_c = \{ c?m, c!m, \overline{c?m}, \overline{c!m} \} \).

b. Construct the asynchronous product \( P \) of the three automata obtained in (a). The alphabet of the automaton \( P \) should be \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_c \).

c. Consider the state of the asynchronous product \( P \) where both processes are in the critical section. Is this state reachable? Give a short justification based on automaton \( P \).

Question 5  (3 points)
Consider the following DBAs \( B_1 \) and \( B_2 \):

a. Give \( \omega \)-regular expressions recognizing the languages of \( B_1 \) and \( B_2 \).

b. Give the DBA \( B_1 \cap B_2 \) using the algorithm seen in class. Give an \( \omega \)-regular expression for \( B_1 \cap B_2 \).
**Question 6** (3 points)
Let $B$ be the following Büchi automaton.

```plaintext
s0  b  
\(a\)  a  
\(s_3\)  \(s_1\)  \(s_2\)  \(b\)  
\(a\)  \(s_4\)  \(s_5\)
```

a. For every state of $B$, give the discovery time and finishing time assigned by a DFS on $B$ starting in $s_0$ (i.e. the moment they first become grey and the moment they become black). Visit successors $s_i$ of a given state in the ascending order of their indices $i$. For example, when visiting the successors of $s_2$, first visit $s_3$ and later $s_4$.

b. The language of $B$ is not empty. Give the witness lasso (as a sequence of states) found by applying `TwoStack` to $B$ following the same convention for the order of successors as above.

**Question 7** (6 points)
Let $\text{AP} = \{p, q\}$ and let $\Sigma = 2^{\text{AP}}$. Recall: An LTL formula is a tautology if it is satisfied by all computations.

a. Is the following formula a tautology: $(\text{GF} p \land \text{GF} q) \Rightarrow \text{G}(p \mathbin{U} q)$? Provide a formal proof if it is and a counter-example if it is not.

b. Is the following formula a tautology: $\text{G}(p \mathbin{U} q) \Rightarrow (\text{GF} p \lor \text{GF} q)$? Provide a formal proof if it is and a counter-example if it is not.

c. Give a Büchi automaton for the $\omega$-language over $\Sigma$ defined by the following LTL formula: $\text{G}(p \mathbin{U} q)$.

**Question 8** (3 points)
★ Given a language $L$ we define the language $\text{Cycle}(L) = \{vu \mid uv \in L\}$. For example, if $L = \{ab, abcd\}$ then $\text{Cycle}(L) = \{ab, ba, abcd, bcda, cdab, dabc\}$; in particular, $abcd$ is not in $\text{Cycle}(L)$ as it cannot be written as $vu$ such that $w \in L$.

Find a language $L$ such that $L$ is not regular and $\text{Cycle}(L)$ is regular. Give proofs for both statements.