Technische Universität München (I7) **Winter Semester 2021-22**
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Quantitative Verification – Exercise sheet 6

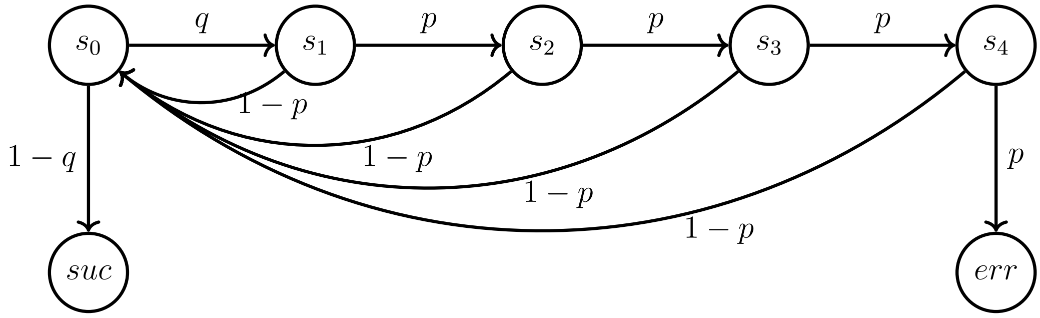
Exercise 6.1
Prove the following statements:

- Let P be a stochastic matrix, i.e. the matrix representation of some Markov Chain. Then, $\frac{1}{2}P + \frac{1}{2}I$ is aperiodic, where I is the unit matrix.
- There exists a finite state Markov Chain and an initial distribution π_0 with a limiting distribution π^*, but $\pi_n = P^n \pi_0 \neq \pi^*$ for all $n \in \mathbb{N}$.
- If all states are irreducible, aperiodic and recurrent non-null in a Markov Chain, there is a unique limiting distribution which does not depend on π_0. Show that each of these properties are required by finding a Markov Chain which
  (i) does not have a unique limiting distribution, and
  (ii) satisfies all but one of the properties.

Exercise 6.2
Imagine jobs arriving at a server with an unbounded queue. The server works on a job at the head of the queue and, when finished, moves on to the next job. The server never drops a job, but just allows them to queue up. At every time step, with probability $p = \frac{1}{50}$ one job arrives, and independently, with probability $q = \frac{1}{30}$ one job departs, i.e. is finished by the server. Note that during a time step, we might have both an arrival and a transmission, or neither. Draw the Markov Chain modelling this server (assuming that you are interested in studying the number of jobs in the system).

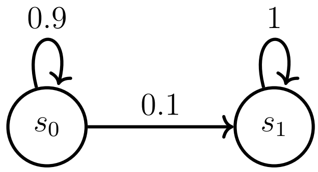
Exercise 6.3
A simplified version of the IPv4 Zeroconf protocol is outlined below. Model the protocol in PRISM and compute the transient probabilities for the first few steps.

1. Randomly pick an address among the K (65024) addresses.
2. With M hosts in the network, collision probability is $q = \frac{M}{K}$
3. Send 4 ARP requests.
4. In case of collision, the probability of no answer to the ARP request is p (due to the lossy channel).


s_0
q
s_1
p
s_2
p
s_3
p
s_4
$1-p$
$1-p$
$1-p$
$1-p$
$1-q$
p
suc
err

Solution 6.1

- For any state, we have $p_{ii}^1 \geq \frac{1}{2}$, hence $d_i = 1$.
- Look at the following Markov Chain:

  With initial distribution $\pi_0 = [1, 0]$, the limiting distribution is $\pi^* = [0, 1]$ but we have $\pi_n = [0.9^n, 1 - 0.9^n]$.

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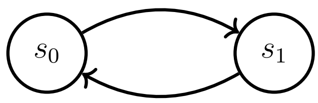
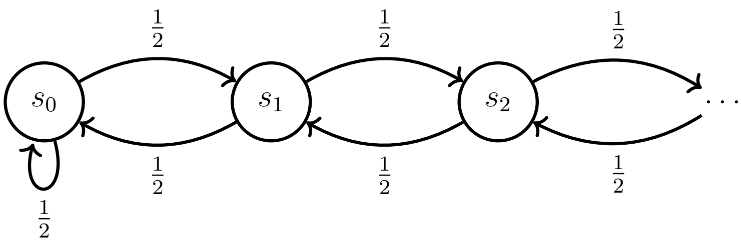
  - **Irreducible:** Each π_0 gives a stationary distribution equal to itself.

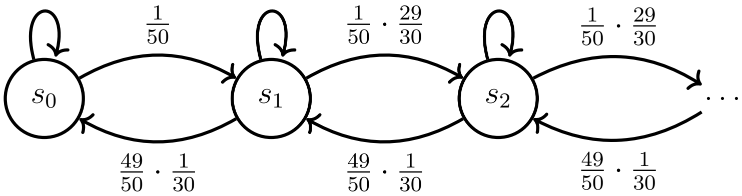
  - **Aperiodic:** Take $\pi_0 = [1, 0]$, then $\pi_1 = [0, 1]$, $\pi_2 = [1, 0]$ ...

  - **Recurrent non-null:** No state is recurrent non-null, all recurrent null. No limiting distribution exists, since π_n converges to 0 point-wise.

Solution 6.2



Solution 6.3

Look at the prism file `zeroconf.pm`. To compute transient probabilities using PRISM GUI, select *Model* > *Compute* > *Transient probabilities*.