Topic:

Semantic Analysis

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$\rightsquigarrow$ a semantic analysis annotates the syntax tree with attributes

Chapter 1:

Attribute Grammars

Attribute Grammars

- many computations of the semantic analysis as well as the code generation operate on the syntax tree
- what is computed at a given node only depends on the *type* of that node (which is usually a non-terminal)
- we call this a *local* computation:
  - only accesses already computed information from neighbouring nodes
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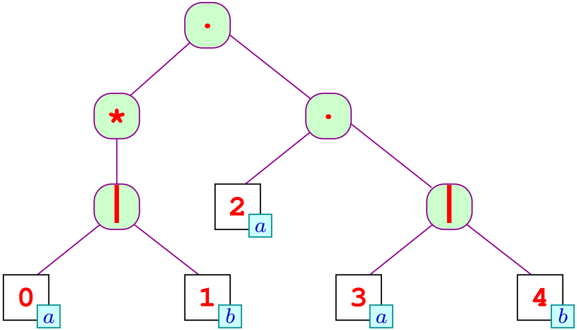
- a set of attributes for each non-terminal and terminal
- local attribute equations

- in order to be able to evaluate the attribute equations, all attributes mentioned in that equation have to be evaluated already
  $\leadsto$ the nodes of the syntax tree need to be visited in a certain *sequence*

Example: Computation of the empty$[r]$ Attribute

Consider the syntax tree of the regular expression (a|b)*a(a|b):



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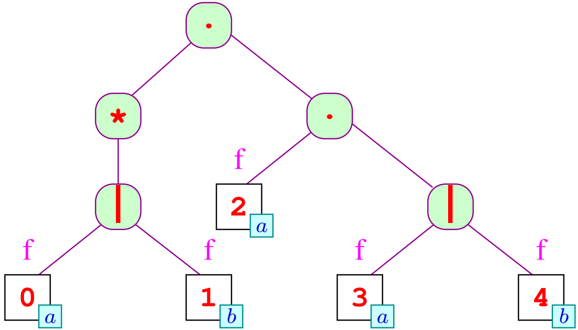

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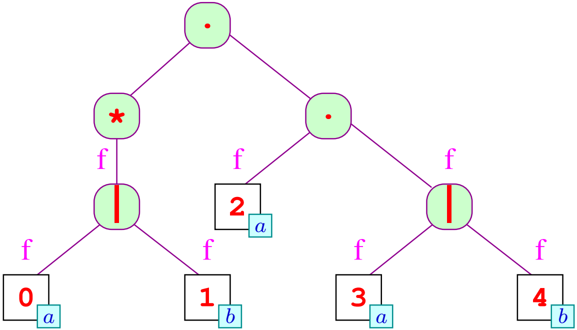

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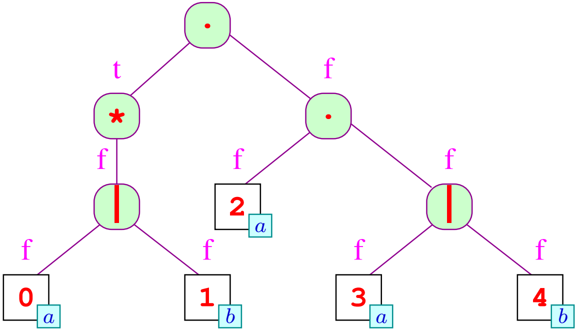

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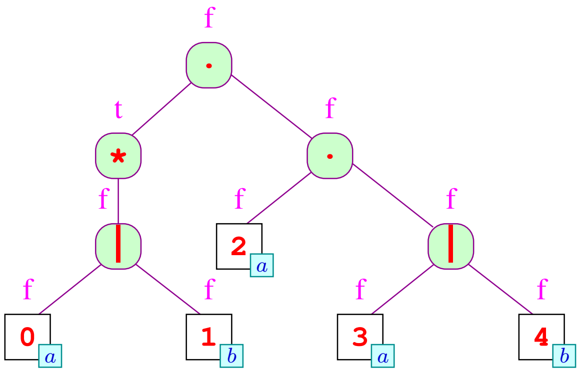

$\rightsquigarrow$ equations for empty$[r]$ are computed from bottom to top (aka bottom-up)

Implementation Strategy

- attach an attribute `empty` to every node of the syntax tree
- compute the attributes in a *depth-first* post-order traversal:
  - at a leaf, we can compute the value of `empty` without considering other nodes
  - the attribute of an inner node only depends on the attribute of its children
- the `empty` attribute is a *synthesized* attribute

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in general:

Definition

An attribute at N is called

- *inherited* if its value is defined in terms of attributes of N's parent, siblings and/or N itself (root $\hookrightarrow$ leaves)
- *synthesized* if its value is defined in terms of attributes of N's children and/or N itself (leaves $\to$ root)

Example: Attribute Equations for `empty`

In order to compute an attribute *locally*, specify attribute equations for each node

Example: Attribute Equations for empty

In order to compute an attribute *locally*, specify attribute equations for each node depending on the *type* of the node:

In the Example from earlier, we did that intuitively:

for leaves: $r \equiv \boxed{i \mid x}$ we define $\text{empty}[r] = (x \equiv \epsilon)$.

otherwise:

$$
\begin{array}{lcl}
\text{empty}[r_1 \mid r_2] & = & \text{empty}[r_1] \vee \text{empty}[r_2] \\
\text{empty}[r_1 \cdot r_2] & = & \text{empty}[r_1] \wedge \text{empty}[r_2] \\
\text{empty}[r_1^*] & = & t \\
\text{empty}[r_1?] & = & t
\end{array}
$$

Specification of General Attribute Systems

General Attribute Systems

In general, for establishing attribute systems we need a flexible way to *refer to parents and children*:

$\rightsquigarrow$ We use consecutive indices to refer to neighbouring attributes

$\mathsf{attribute}_k[0]$:	the attribute of the current root node
$\mathsf{attribute}_k[i]$:	the attribute of the i-th child $\quad (i > 0)$

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... the example, now in general formalization:

$$
\begin{array}{llll}
\boxed{x} & : & \text{empty}[0] & := (x \equiv \epsilon) \\
\boxed{\,|\,} & : & \text{empty}[0] & := \text{empty}[1] \vee \text{empty}[2] \\
\boxed{\cdot} & : & \text{empty}[0] & := \text{empty}[1] \wedge \text{empty}[2] \\
\boxed{*} & : & \text{empty}[0] & := t \\
\boxed{?} & : & \text{empty}[0] & := t
\end{array}
$$

Observations

- the *local* attribute equations need to be evaluated using a *global* algorithm that knows about the dependencies of the equations
- in order to construct this algorithm, we need
  1. a sequence in which the nodes of the tree are visited
  2. a sequence within each node in which the equations are evaluated
- this *evaluation strategy* has to be compatible with the *dependencies* between attributes

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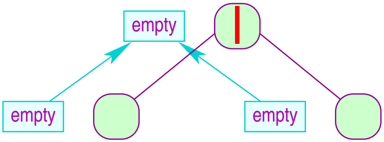
We visualize the attribute dependencies $D(p)$ of a production p in a *Local Dependency Graph*:



Let $p = N_0 \mapsto N_1 | N_2$ in

$$D(p) = \{ \quad (empty[1], empty[0]), \\ (empty[2], empty[0])\}$$

$\leadsto$ arrows point in the direction of information flow

Observations

- in order to infer an evaluation strategy, it is not enough to consider the *local* attribute dependencies at each node
- the evaluation strategy must also depend on the *global* dependencies, that is, on the information flow between nodes

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- in order to infer an evaluation strategy, it is not enough to consider the *local* attribute dependencies at each node
- the evaluation strategy must also depend on the *global* dependencies, that is, on the information flow between nodes
- ⚠ the global dependencies change with each particular syntax tree
- in the example, the parent node is always depending on children only
  $\leadsto$ a depth-first post-order traversal is possible
- in general, variable dependencies can be much *more complex*

Simultaneous Computation of Multiple Attributes

Computing empty, first, next from regular expressions:

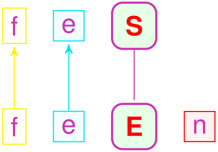
$\boxed{S \rightarrow E:}$:

$$\begin{array}{lcl} \text{empty}[0] & := & \text{empty}[1] \\ \text{first}[0] & := & \text{first}[1] \\ \text{next}[1] & := & \emptyset \end{array}$$

$\boxed{E \rightarrow x}$:

$$\begin{array}{lcl} \text{empty}[0] & := & (x \equiv \epsilon) \\ \text{first}[0] & := & \{x \mid x \neq \epsilon\} \end{array}$$

$D(S \rightarrow E)$:



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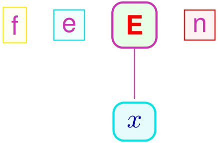

$$D(S \rightarrow E) = \{ \quad (empty[1], empty[0]), \\ (first[1], first[0])\}$$

$$D(E \rightarrow x) = \{ \quad \}$$

Regular Expressions: Rules for Alternative

$\boxed{E \rightarrow E|E}$:

$$\begin{array}{lcl} \mathsf{empty}[0] & := & \mathsf{empty}[1] \vee \mathsf{empty}[2] \\ \mathsf{first}[0] & := & \mathsf{first}[1] \cup \mathsf{first}[2] \\ \mathsf{next}[1] & := & \mathsf{next}[0] \\ \mathsf{next}[2] & := & \mathsf{next}[0] \end{array}$$

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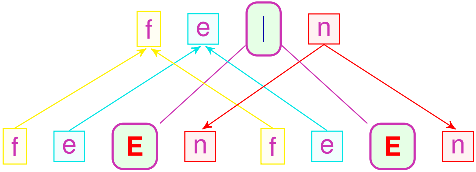

$$D(E \rightarrow E|E) = \{ \begin{array}{l} (empty[1], empty[0]), \\ (empty[2], empty[0]), \\ (first[1], first[0]), \\ (first[2], first[0]), \\ (next[0], next[2]), \\ (next[0], next[1]) \} \end{array}$$

Regular Expressions: Rules for Concatenation

$\boxed{E \to E \cdot E}$:

$$
\begin{aligned}
\mathsf{empty}[0] &:= \mathsf{empty}[1] \wedge \mathsf{empty}[2] \\
\mathsf{first}[0] &:= \mathsf{first}[1] \cup (\mathsf{empty}[1] \;?\; \mathsf{first}[2] : \emptyset) \\
\mathsf{next}[1] &:= \mathsf{first}[2] \cup (\mathsf{empty}[2] \;?\; \mathsf{next}[0] : \emptyset) \\
\mathsf{next}[2] &:= \mathsf{next}[0]
\end{aligned}
$$

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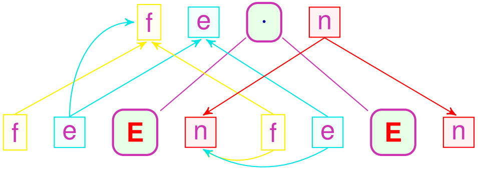

$$
\begin{aligned}
D(E \to E \cdot E) = \{ \quad & (empty[1], empty[0]), \\
& (empty[2], empty[0]), \\
& (empty[2], next[1]), \\
& (empty[1], first[0]), \\
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Regular Expressions: Rules for Kleene-Star and Option

$\boxed{E \to E*}$:

$$\begin{aligned} \text{empty}[0] &:= t \\ \text{first}[0] &:= \text{first}[1] \\ \text{next}[1] &:= \text{first}[1] \cup \text{next}[0] \end{aligned}$$

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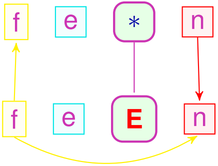

$$D(E \to E*) = \{ \; (first[1], first[0]), \; (first[1], next[2]), \; (next[0], next[1]) \}$$

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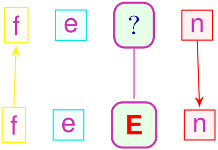

$$D(E \to E?) = \{ \; (first[1], first[0]), \; (next[0], next[1]) \}$$

Challenges for General Attribute Systems

Static evaluation

Is there a static evaluation strategy, which is generally applicable?

- an evaluation strategy can only exist, if for *any* derivation tree the dependencies between attributes are acyclic
- it is *DEXPTIME*-complete to check for cyclic dependencies [Jazayeri, Odgen, Rounds, 1975]

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Ideas

1. Let the *User* specify the strategy
2. Determine the strategy dynamically
3. Automate *subclasses* only

Subclass: Strongly Acyclic Attribute Dependencies

Idea: For all nonterminals X compute a set $\mathcal{R}(X)$ of relations between its attributes, as an *overapproximation of the global dependencies* between root attributes of every production for X.

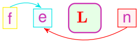
Describe $\mathcal{R}(X)$s as sets of relations, similar to $D(p)$ by

- setting up each production $X \mapsto X_1 \ldots X_k$'s effect on the relations of $\mathcal{R}(X)$
- compute effect on all so far accumulated evaluations of each rhs X_i's $\mathcal{R}(X_i)$
- iterate until stable

Subclass: Strongly Acyclic Attribute Dependencies

The 2-ary operator $L[i]$ re-decorates relations from L

$$L[i] = \{(a[i], b[i]) \mid (a, b) \in L\}$$



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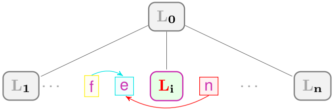

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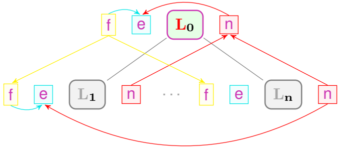
$$\pi_0(S) = \{(a, b) \mid (a[0], b[0]) \in S\}$$



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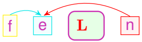

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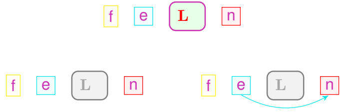

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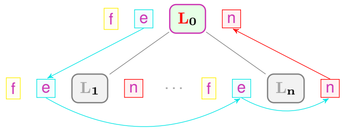

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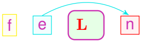

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$\mathcal{R}$ maps symbols to relations (global attributes dependencies)

$$\mathcal{R}(X) \supseteq (\bigcup\{[\![p]\!]^\sharp(\mathcal{R}(X_1), \ldots, \mathcal{R}(X_k)) \mid p : X \to X_1 \ldots X_k\})^+ \mid p \in P$$

$$\mathcal{R}(X) \supseteq \emptyset \quad \mid X \in (N \cup T)$$

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Strongly Acyclic Grammars

The system of inequalities $\mathcal{R}(X)$

- characterizes the class of strongly acyclic Dependencies
- has a unique least solution $\mathcal{R}^\star(X)$ (as $[\![.]\!]^\sharp$ is monotonic)

Subclass: Strongly Acyclic Attribute Dependencies

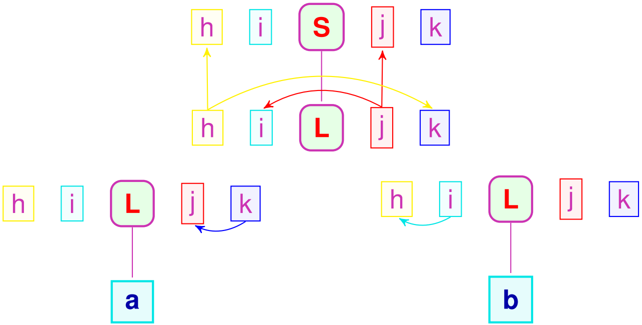
Strongly Acyclic Grammars

If all $D(p) \cup \mathcal{R}^\star(X_1)[1] \cup \ldots \cup \mathcal{R}^\star(X_k)[k]$ are acyclic for all $p \in G$, G is strongly acyclic.

Idea: we compute the least solution $\mathcal{R}^\star(X)$ of $\mathcal{R}(X)$ by a fixpoint computation, starting from $\mathcal{R}(X) = \emptyset$.

Example: Strong Acyclic Test

Given grammar $S \rightarrow L$, $L \rightarrow a \mid b$. Dependency graphs D_p:



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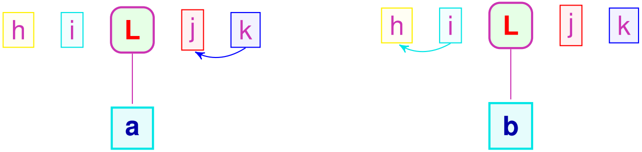
Start with computing $\mathcal{R}(L) = [\![L \to a]\!]^\sharp() \sqcup [\![L \to b]\!]^\sharp()$:



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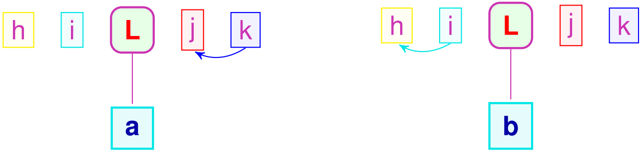


1. terminal symbols do not contribute dependencies

check for cycles!

2. transitive closure of all relations in $(D(L \to a))^+$ and $(D(L \to b))^+$

Example: Strong Acyclic Test

Start with computing $\mathcal{R}(L) = [\![L{\to}a]\!]^\sharp() \sqcup [\![L{\to}b]\!]^\sharp()$:

1. terminal symbols do not contribute dependencies
2. transitive closure of all relations in $(D(L{\to}a))^+$ and $(D(L{\to}b))^+$
3. apply π_0

Example: Strong Acyclic Test

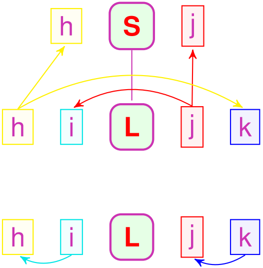
Start with computing $\mathcal{R}(L) = [\![L \to a]\!]^{\sharp}() \sqcup [\![L \to b]\!]^{\sharp}()$:

$$\begin{tikzcd} \boxed{h} & \boxed{i} \arrow[l, bend left] & \boxed{\mathbf{L}} & \boxed{j} & \boxed{k} \arrow[l, bend left] \end{tikzcd}$$

1. terminal symbols do not contribute dependencies
2. transitive closure of all relations in $(D(L \to a))^{+}$ and $(D(L \to b))^{+}$
3. apply π_0
4. $\mathcal{R}(L) = \{(k, j), (i, h)\}$

Example: Strong Acyclic Test

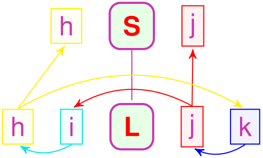
Continue with $\mathcal{R}(S) = [\![S \to L]\!]^\sharp(\mathcal{R}(L))$:



1. re-decorate and embed $\mathcal{R}(L)[1]$

Example: Strong Acyclic Test

Continue with $\mathcal{R}(S) = [\![S \rightarrow L]\!]^\sharp(\mathcal{R}(L))$:



1. re-decorate and embed $\mathcal{R}(L)[1]$

check for cycles!

2. transitive closure of all relations $(D(S \rightarrow L) \cup \{(k[1], j[1])\} \cup \{(i[1], h[1])\})^+$

Example: Strong Acyclic Test

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h $\boxed{\text{S}}$ j

1. re-decorate and embed $\mathcal{R}(L)[1]$
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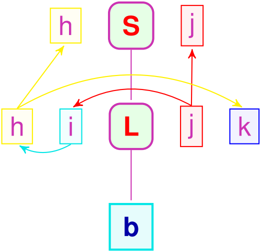
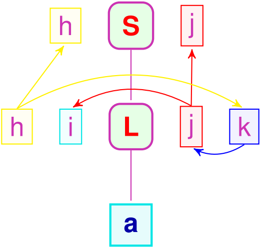
2. transitive closure of all relations $(D(S \rightarrow L) \cup \{(k[1], j[1])\} \cup \{(i[1], h[1])\})^+$
3. apply π_0
4. $\mathcal{R}(S) = \{\}$

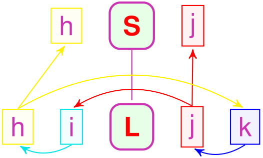
Strong Acyclic and Acyclic

The grammar $S \rightarrow L$, $L \rightarrow a \mid b$ has only two derivation trees which are both *acyclic*:





It is *not strongly acyclic* since the over-approximated global dependence graph for the non-terminal L contributes to a cycle when computing $\mathcal{R}(S)$:



From Dependencies to Evaluation Strategies

Possible strategies:

1. let the *user* define the evaluation order

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From Dependencies to Evaluation Strategies

Possible strategies:

1. let the *user* define the evaluation order
2. *automatic* strategy based on the dependencies
3. consider a *fixed* strategy and only allow an attribute system that can be evaluated using this strategy

Linear Order from Dependency Partial Order

Possible *automatic* strategies:

Linear Order from Dependency Partial Order

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1. demand-driven evaluation
   - start with the evaluation of any required attribute
   - if the equation for this attribute relies on as-of-yet unevaluated attributes, evaluate these recursively

Linear Order from Dependency Partial Order

Possible *automatic* strategies:

1. demand-driven evaluation
   - start with the evaluation of any required attribute
   - if the equation for this attribute relies on as-of-yet unevaluated attributes, evaluate these recursively
2. evaluation in passes
   for each pass, pre-compute a global strategy to visit the *nodes* together with a local strategy for evaluation *within each node* type

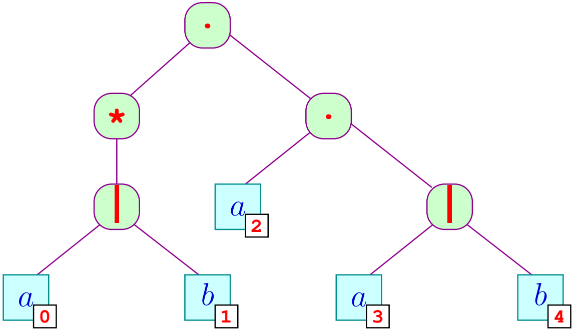
   $\rightsquigarrow$ *minimize* the number of *visits* to each node

Example: Demand-Driven Evaluation

Compute next at leaves a_2, a_3 and b_4 in the expression $(a|b)^* a(a|b)$:

$$\boxed{|} \quad : \quad \begin{aligned} \text{next}[1] &:= \text{next}[0] \\ \text{next}[2] &:= \text{next}[0] \end{aligned}$$

$$\boxed{\cdot} \quad : \quad \begin{aligned} \text{next}[1] &:= \text{first}[2] \cup (\text{empty}[2]\ ?\ \text{next}[0] : \emptyset) \\ \text{next}[2] &:= \text{next}[0] \end{aligned}$$



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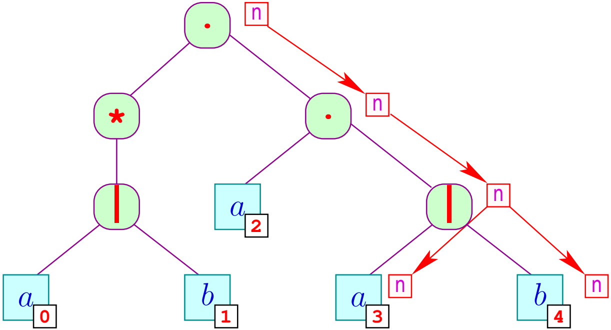

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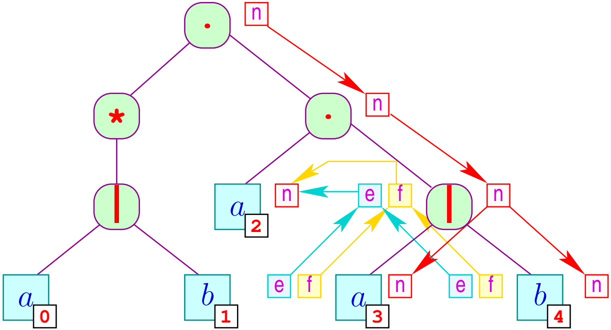

Demand-Driven Evaluation

Observations

- each node must contain a pointer to its parent
- *only required* attributes are evaluated
- the evaluation sequence depends – in general – on the actual syntax tree
- the algorithm must track which attributes it has already evaluated
- the algorithm may visit nodes more often than necessary

$\leadsto$ the algorithm is not local

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- evaluation strategy is dynamic: difficult to debug
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in principle:

- evaluation strategy is dynamic: difficult to debug
- usually all attributes in all nodes are required

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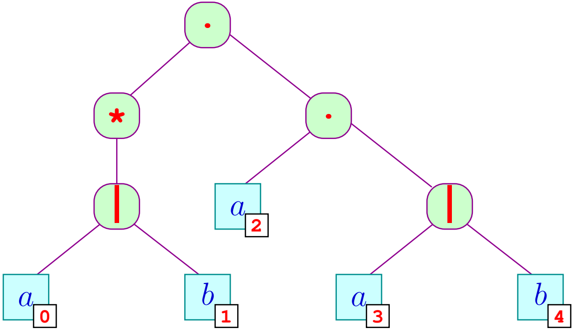
$\leadsto$ perform evaluation in *passes*

Implementing State

Problem: In many cases some sort of state is required.

Example: numbering the leafs of a syntax tree



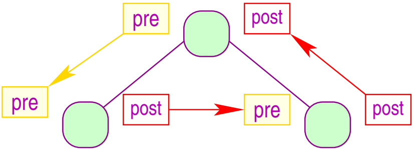
Example: Implementing Numbering of Leafs

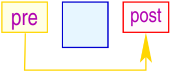
Idea:

- use helper attributes pre and post
- in pre we pass the value for the first leaf down (inherited attribute)
- in post we pass the value of the last leaf up (synthesized attribute)

$$
\begin{array}{llcl}
\text{root:} & \text{pre}[0] & := & 0 \\
 & \text{pre}[1] & := & \text{pre}[0] \\
 & \text{post}[0] & := & \text{post}[1] \\
 & & & \\
\text{node:} & \text{pre}[1] & := & \text{pre}[0] \\
 & \text{pre}[2] & := & \text{post}[1] \\
 & \text{post}[0] & := & \text{post}[2] \\
 & & & \\
\text{leaf:} & \text{post}[0] & := & \text{pre}[0] + 1
\end{array}
$$

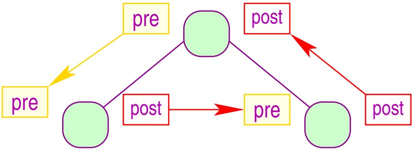
L-Attributation

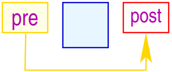




- the attribute system is apparently strongly acyclic

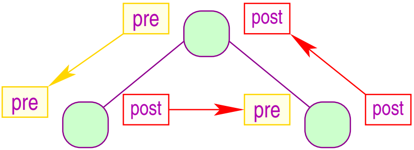
L-Attributation

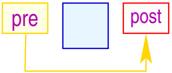




- the attribute system is apparently strongly acyclic
- each node computes
  - the inherited attributes before descending into a child node (corresponding to a pre-order traversal)
  - the synthesized attributes after returning from a child node (corresponding to post-order traversal)

L-Attributation





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  - the inherited attributes before descending into a child node (corresponding to a pre-order traversal)
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Definition L-Attributed Grammars

An attribute system is L-attributed, if for all productions $S \to S_1 \dots S_n$ every inherited attribute of S_j where $1 \leq j \leq n$ only depends on

1. the attributes of S_1, S_2, $\dots S_{j-1}$ and
2. the inherited attributes of S.

L-Attributation

Background:

- the attributes of an L-attributed grammar can be evaluated during parsing
- important if no syntax tree is required or if error messages should be emitted while parsing
- example: pocket calculator

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- the attributes of an L-attributed grammar can be evaluated during parsing
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- example: pocket calculator

L-attributed grammars have a fixed evaluation strategy:
a single *depth-first* traversal

- in general: partition all attributes into $\mathcal{A} = A_1 \cup \ldots \cup A_n$ such that for all attributes in A_i the attribute system is L-attributed
- perform a depth-first traversal for each attribute set A_i

$\leadsto$ craft attribute system in a way that they can be partitioned into few L-attributed sets

Practical Applications

- symbol tables, type checking/inference, and simple code generation can all be specified using L-attributed grammars

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$\leadsto$ the different types of non-terminals are characterized by the set of attributes with which they are decorated

Example: Def-Use Analysis

- *a statement* may have two attributes containing valid identifiers: one ingoing (inherited) set and one outgoing (synthesised) set
- *an expression* only has an ingoing set

Implementation of Attribute Systems via a *Visitor*

- class with a method for every non-terminal in the grammar

```
public abstract class Regex {
  public abstract void accept(Visitor v);
}
```

- attribute-evaluation works via *pre-order / post-order callbacks*

```
public interface Visitor {
  default void pre(OrEx re)   {}
  default void pre(AndEx re)  {}
  ...
  default void post(OrEx re)  {}
  default void post(AndEx re){}
}
```

- we pre-define a depth-first traversal of the syntax tree

```
public class OrEx extends Regex {
  Regex l,r;
  public void accept(Visitor v) {
     v.pre(this);l.accept(v);v.inter(this);
     r.accept(v); v.post(this);
} }
```

Example: Leaf Numbering

```
public abstract class AbstractVisitor implements Visitor {
  public void pre (OrEx  re){ pr(re); }
  public void pre (AndEx re){ pr(re); }
  ... /* redirecting to default handler for bin exprs */
  public void post(OrEx  re){ po(re); }
  public void post(AndEx re){ po(re); }
  abstract void po(BinEx re);
  abstract void in(BinEx re);
  abstract void pr(BinEx re);
}
public class LeafNum extends AbstractVisitor {
  public Map<Regex,Integer> pre  = new HashMap<>();
  public Map<Regex,Integer> post = new HashMap<>();
  public LeafNum (Regex r) { pre .put(r,0); r.accept(this); }
  public void pre(Const r) { post.put(r,    pre .get(r)+1); }
  public void pr (BinEx r) { pre .put(r.l,  pre .get(r));   }
  public void in (BinEx r) { pre .put(r.r,  post.get(r.l)); }
  public void po (BinEx r) { post.put(r,    post.get(r.r)); }
}
```

Chapter 2:

Decl-Use Analysis

Symbol Bindings and Visibility

Consider the following Java code:

```
void foo() {
  int a;
  while(true) {
    double a;
    a = 0.5;
    write(a);
    break;
  }
  a = 2;
  bar();
  write(a);
}
```

- each *declaration* of a variable v causes memory allocation for v
- using v requires knowledge about its memory location
  $\rightarrow$ determine the declaration v is *bound* to
- a binding is not *visible* when a local declaration of the same name is in scope

  in the example the definition of `A` is shadowed by the *local definition* in the loop body

Scope of Identifiers

```
void foo() {
  int A;
  while (true) {
    double A;
    A = 0.5;
    write(A);
    break;
  }
  A = 2;
  bar();
  write(A);
}
```

scope of **int** A

Scope of Identifiers

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void foo() {
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```

scope of **double** A

Scope of Identifiers

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    double A;
    A = 0.5;
    write(A);
    break;
  }
  A = 2;
  bar();
  write(A);
}
```

scope of **double** A

administration of identifiers can be quite complicated...

Resolving Identifiers

Observation: each identifier in the AST must be translated into a memory access

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Problem: for each identifier, find out what memory needs to be accessed by providing *rapid* access to its *declaration*

Idea:

1. *rapid* access: replace every identifier by a *unique* integer
   $\rightarrow$ integers as keys: comparisons of integers is faster

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Idea:

1. *rapid* access: replace every identifier by a *unique* integer
   $\rightarrow$ integers as keys: comparisons of integers is faster
2. link each usage of a variable to the *declaration* of that variable
   $\rightarrow$ for languages without explicit declarations, create declarations when a variable is first encountered

Rapid Access: Replace Strings with Integers

Idea for Algorithm:

Input: a sequence of strings

Output:
1. sequence of numbers
2. table that allows to retrieve the string that corresponds to a number

Apply this algorithm on each identifier during *scanning*.

Implementation approach:

- count the number of new-found identifiers in $\mathbf{int}\ \mathsf{count}$
- maintain a *hashtable* $S : \mathbf{String} \to \mathbf{int}$ to remember numbers for known identifiers

We thus define the function:

$$
\begin{array}{l}
\mathbf{int}\ \mathsf{indexForIdentifier}(\mathbf{String}\ w)\ \{ \\
\quad \mathbf{if}\ (S\,(w)\ \equiv\ \mathsf{undefined})\ \{ \\
\qquad\quad S = S \oplus \{w \mapsto \mathsf{count}\}; \\
\qquad\quad \mathbf{return}\ \ \mathsf{count}{+}{+}; \\
\quad \}\ \mathbf{else}\ \ \mathbf{return}\ \ S\,(w); \\
\}
\end{array}
$$

Implementation: Hashtables for Strings

1. allocate an array M of sufficient size m
2. choose a *hash function* $H : \mathbf{String} \to [0, m-1]$ with:
   - $H(w)$ is cheap to compute
   - H distributes the occurring words equally over $[0, m-1]$

Possible generic choices for sequence types ($\vec{x} = \langle x_0, \ldots x_{r-1} \rangle$):

$$\begin{aligned} H_0(\vec{x}) &= (x_0 + x_{r-1}) \,\%\, m \\ H_1(\vec{x}) &= \left(\sum_{i=0}^{r-1} x_i \cdot p^i\right) \%\, m \\ &= (x_0 + p \cdot (x_1 + p \cdot (\ldots + p \cdot x_{r-1} \cdots))) \,\%\, m \end{aligned}$$

for some prime number p (e.g. 31)

✗ The hash value of w *may not be unique*!

→ Append (w, i) to a linked list located at $M[H(w)]$

- Finding the index for w, we compare w with all x for which $H(w) = H(x)$

✓ access on average:

insert: $\mathcal{O}(1)$
lookup: $\mathcal{O}(1)$

Example: Replacing Strings with Integers

Input:

Peter	Piper	picked	a	peck	of	pickled	peppers

If	Peter	Piper	picked	a	peck	of	pickled	peppers

wheres	the	peck	of	pickled	peppers	Peter	Piper	picked

Output:

Example: Replacing Strings with Integers

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If	Peter	Piper	picked	a	peck	of	pickled	peppers

wheres	the	peck	of	pickled	peppers	Peter	Piper	picked

Output:

0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6

7	9	10	4	5	6	7	0	1	2

Example: Replacing Strings with Integers

Input:

Peter	Piper	picked	a	peck	of	pickled	peppers	
If	Peter	Piper	picked	a	peck	of	pickled	peppers
wheres	the	peck	of	pickled	peppers	Peter	Piper	picked

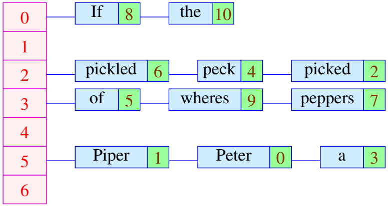
Output:

0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6
7	9	10	4	5	6	7	0	1	2						

and

0	Peter
1	Piper
2	picked
3	a
4	peck
5	of
6	pickled
7	peppers
8	If
9	wheres
10	the

Hashtable with $m = 7$ and H_0:



Refer Uses to Declarations: Symbol Tables

Check for the correct usage of variables:

- Traverse the syntax tree in a suitable sequence, such that
  - each declaration is visited before its use
  - the currently visible declaration is the last one visited

  $\leadsto$ perfect for an L-attributed grammar
  - equation system for basic block must add and remove identifiers
- for each identifier, we manage a *stack* of declarations
  1. if we visit a *declaration*, we push it onto the stack of its identifier
  2. upon leaving the *scope*, we remove it from the stack
- if we visit a *usage* of an identifier, we pick the top-most declaration from its stack
- if the stack of the identifier is empty, we have found an undeclared identifier

Example: Decl-Use Analysis via Table of Stacks

```
void f()
{                              ⇐
  int a, b;
  b = 5;
  if (b>3) {
    int a, c;
    a = 3;
    c = a + 1;
    b = c;
  } else {
    int c;
    c = a + 1;
    b = c;
  }
  b = a + b;
}
```

0	a
1	b
2	c

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  b = a + b;
}
```

0	a
1	b
2	c

$\boxed{3}$

$\boxed{3}$

Example: Decl-Use Analysis via Table of Stacks

```c
void f()
{
  int a, b;
  b = 5;
  if (b>3) {
    int a, c;
    a = 3;
    c = a + 1;
    b = c;
  } else {
    int c;
    c = a + 1;
    b = c;
  }
  b = a + b;
}
```

0	a
1	b
2	c

6	3
	3
	6

Example: Decl-Use Analysis via Table of Stacks

```
void f()
{
   int a, b;
   b = 5;
   if (b>3) {
      int a, c;
      a = 3;
      c = a + 1;
      b = c;
   } else {
      int c;
      c = a + 1;   ⇐
      b = c;
   }
   b = a + b;
}
```

0	a	3
1	b	3
2	c	11

Example: Decl-Use Analysis via Table of Stacks

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      b = c;
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      b = c;
   }
   b = a + b;      ⇐
}
```

0	a
1	b
2	c

3
3

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}
```

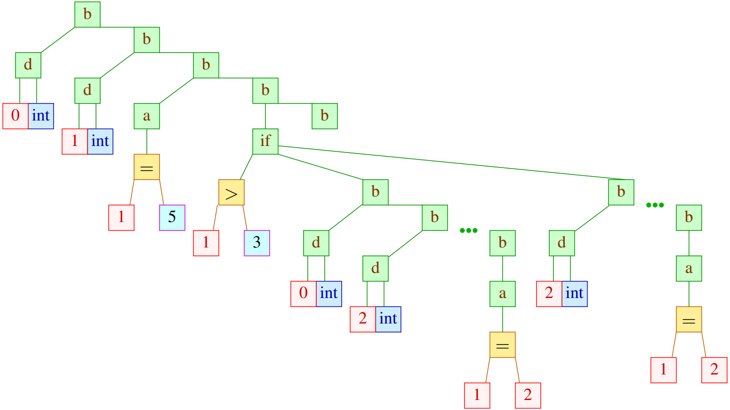
Example: Decl-Use Analysis via Table of Stacks

```
void f()
{
    int a, b;
    b = 5;
    if (b>3) {
        int a, c;
        a = 3;
        c = a + 1;
        b = c;
    } else {
        int c;
        c = a + 1;
        b = c;
    }
    b = a + b;
}
```

d declaration

b basic block

a assignment



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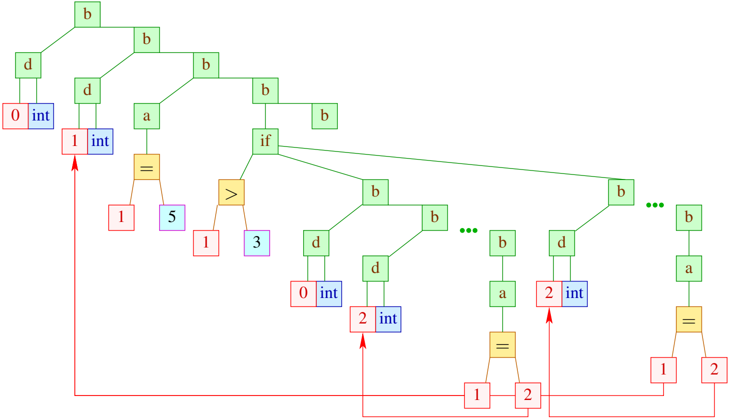

Alternative Implementations for Symbol Tables

- when using a list to store the symbol table, storing a marker indicating the old head of the list is sufficient

a
b

in front of if-statement

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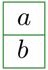
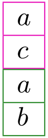


in front of if-statement



then-branch

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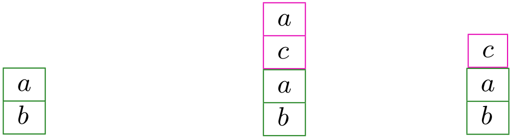

in front of if-statement | then-branch | else-branch

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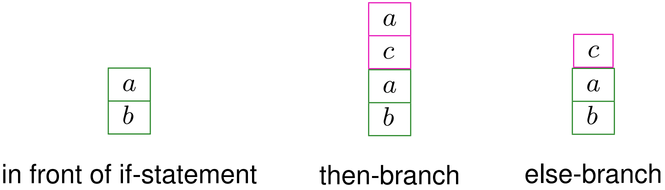

- instead of lists of symbols, it is possible to use a list of hash tables $\leadsto$ more efficient in large, shallow programs

Alternative Implementations for Symbol Tables

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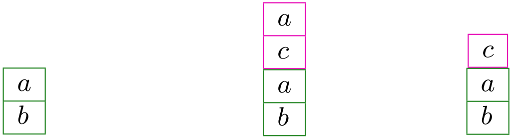

in front of if-statement　　then-branch　　else-branch

- instead of lists of symbols, it is possible to use a list of hash tables $\leadsto$ more efficient in large, shallow programs
- an even more elegant solution: *persistent trees* (updates return fresh trees with references to the old tree where possible)

$\leadsto$ a persistent tree t can be passed down into a basic block where new elements may be added, yielding a t'; after examining the basic block, the analysis proceeds with the unchanged old t

Chapter 3:

Type Checking

Goal of Type Checking

In most mainstream (imperative / object oriented / functional) programming languages, variables and functions have a fixed type.
for example: `int`, `void*`, `struct { int x; int y; }`.

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Types are useful to

- manage memory
- to avoid certain run-time errors

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Types are useful to

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In imperative and object-oriented programming languages a declaration has to specify a type. The compiler then checks for a type correct use of the declared entity.

Type Expressions

Types are given using type-*expressions*.
The set of type expressions T contains:

1. base types: **`int`**, **`char`**, **`float`**, **`void`**, ...
2. type constructors that can be applied to other types

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2. type constructors that can be applied to other types

example for type constructors in C:

- structures: **struct** $\{\ t_1\ a_1; \ldots t_k\ a_k;\ \}$
- pointers: $t\ *$
- arrays: t [ ]
  - the size of an array can be specified
  - the variable to be declared is written between t and $[n]$
- functions: $t\ (t_1, \ldots, t_k)$
  - the variable to be declared is written between t and $(t_1, \ldots, t_k)$
  - in ML function types are written as: $t_1 * \ldots * t_k \to t$

Type Checking

Problem:

Given: A set of type declarations $\Gamma = \{t_1\ x_1; \ldots t_m\ x_m;\}$
Check: Can an expression e be given the type t?

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Check: Can an expression e be given the type t?

Example:

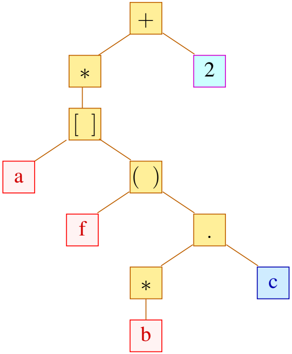
```
struct list { int info; struct list* next; };
int f(struct list* l) { return 1; };
struct { struct list* c;}* b;
int* a[11];
```

Consider the expression:

```
*a[f(b->c)]+2;
```

Type Checking using the Syntax Tree

Check the expression `*a[f(b->c)]+2`:



Idea:

- traverse the syntax tree bottom-up
- for each identifier, we lookup its type in Γ
- constants such as 2 or 0.5 have a fixed type
- the types of the inner nodes of the tree are deduced using *typing rules*

Type Systems

Formally: consider *judgements* of the form:

$$\Gamma \vdash e \; : \; t$$

// (in the type environment Γ the expression e has type t)

Axioms:

Const: $\Gamma \vdash c \; : \; t_c$ (t_c type of constant c)
Var: $\Gamma \vdash x \; : \; \Gamma(x)$ (x Variable)

Rules:

Ref: $$\frac{\Gamma \vdash e \; : \; t}{\Gamma \vdash \& e \; : \; t*}$$

Deref: $$\frac{\Gamma \vdash e \; : \; t*}{\Gamma \vdash *e \; : \; t}$$

Type Systems for C-like Languages

More rules for typing an expression:

Array:
$$\frac{\Gamma \vdash e_1 : t* \qquad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1[e_2] : t}$$

Array:
$$\frac{\Gamma \vdash e_1 : t\,[\,] \qquad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1[e_2] : t}$$

Struct:
$$\frac{\Gamma \vdash e : \mathbf{struct}\ \{t_1\ a_1; \ldots t_m\ a_m;\}}{\Gamma \vdash e.a_i : t_i}$$

App:
$$\frac{\Gamma \vdash e : t\,(t_1, \ldots, t_m) \qquad \Gamma \vdash e_1 : t_1 \quad \ldots \quad \Gamma \vdash e_m : t_m}{\Gamma \vdash e(e_1, \ldots, e_m) : t}$$

Op $\Box$:
$$\frac{\Gamma \vdash e_1 : t \qquad \Gamma \vdash e_2 : t}{\Gamma \vdash e_1 \Box e_2 : t}$$

Op $=$:
$$\frac{\Gamma \vdash e_1 : t_1 \qquad \Gamma \vdash e_2 : t_2 \qquad t_2 \text{ can be converted to } t_1}{\Gamma \vdash e_1 = e_2 : t_1}$$

Explicit Cast:
$$\frac{\Gamma \vdash e : t_2 \qquad t_2 \text{ can be converted to } t_1}{\Gamma \vdash (t_1)\ e : t_1}$$

Type Systems for C-like Languages

More rules for typing an expression: with subtyping relation $\leq$

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Array: $$\frac{\Gamma \vdash e_1 : t[\,] \qquad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1[e_2] : t}$$

Struct: $$\frac{\Gamma \vdash e : \mathbf{struct}\ \{t_1\ a_1; \ldots t_m\ a_m;\}}{\Gamma \vdash e.a_i : t_i}$$

App: $$\frac{\Gamma \vdash e : t\,(t_1, \ldots, t_m) \qquad \Gamma \vdash e_1 : t_1 \quad \ldots \quad \Gamma \vdash e_m : t_m}{\Gamma \vdash e(e_1, \ldots, e_m) : t}$$

Op $\Box$: $$\frac{\Gamma \vdash e_1 : t_1 \qquad \Gamma \vdash e_2 : t_2}{\Gamma \vdash e_1 \Box e_2 : t_1 \sqcup t_2}$$

Op $=$: $$\frac{\Gamma \vdash e_1 : t_1 \qquad \Gamma \vdash e_2 : t_2 \qquad t_2 \leq t_1}{\Gamma \vdash e_1 = e_2 : t_1}$$

Explicit Cast: $$\frac{\Gamma \vdash e : t_2 \qquad t_2 \leq t_1}{\Gamma \vdash (t_1)\ e : t_1}$$

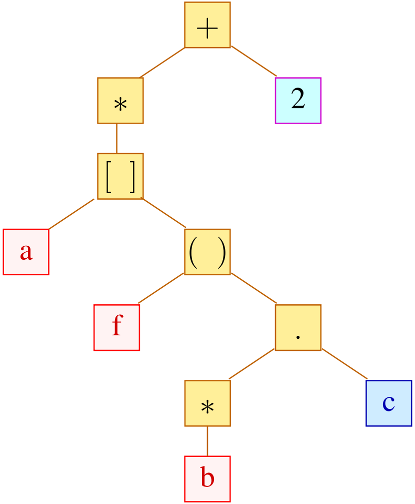
Example: Type Checking

Given expression `*a[f(b->c)]+2` and

$\Gamma = \{$

```
struct list { int info; struct list* next; };
int f(struct list* l);
struct { struct list* c;}* b;
int* a[11];
```

$\}$



Example: Type Checking – More formally:

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struct list { int info; struct list* next; };
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$$\textsc{Op}\ \dfrac{\textsc{Deref}\ \dfrac{\Gamma \vdash a[f(b \to c)] :}{\Gamma \vdash *a[f(b \to c)] : t} \qquad \textsc{Const}\ \dfrac{}{\Gamma \vdash 2 : t}}{\Gamma \vdash *a[f(b \to c)] + 2 : t}$$

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$$

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$$\dfrac{\dfrac{}{\Gamma \vdash a :}\ \textsc{Var} \qquad \dfrac{\dfrac{}{\Gamma \vdash f : \_(t)}\ \textsc{Var} \qquad \Gamma \vdash (*b).c : t}{\Gamma \vdash f(b \to c) : \text{int}}\ \textsc{App}}{\Gamma \vdash a[f(b \to c)] :}\ \textsc{Array}$$

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$$\text{ARRAY}\ \dfrac{\text{VAR}\ \dfrac{}{\Gamma \vdash a :} \qquad \text{APP}\ \dfrac{\text{VAR}\ \dfrac{}{\Gamma \vdash f : \_(\text{struct list}*)} \qquad \Gamma \vdash (*b).c : \ \text{struct list}*}{\Gamma \vdash f(b \to c) : \text{int}}}{\Gamma \vdash a[f(b \to c)] :}$$

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$$\textsc{Array}\ \dfrac{\textsc{Var}\ \dfrac{}{\Gamma \vdash a : \text{int}*[]} \qquad \textsc{App}\ \dfrac{\textsc{Var}\ \dfrac{}{\Gamma \vdash f : \text{int(struct list*)}\ \checkmark} \qquad \Gamma \vdash (*b).c : \ \text{struct list}*}{\Gamma \vdash f(b \to c) : \text{int}\ \checkmark}}{\Gamma \vdash a[f(b \to c)] :}$$

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$\}$

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$$\text{OP}\;\dfrac{\text{DEREF}\;\dfrac{\Gamma \vdash a[f(b \to c)] : \text{int}*}{\Gamma \vdash *a[f(b \to c)] : \text{int}} \qquad \text{CONST}\;\dfrac{}{\Gamma \vdash 2 : \text{int}\ \checkmark}}{\Gamma \vdash *a[f(b \to c)] + 2 : \text{int}}$$

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$\}$

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$$\textsc{Op}\ \dfrac{\textsc{Deref}\ \dfrac{\Gamma \vdash a[f(b \to c)] : \text{int}*}{\Gamma \vdash *a[f(b \to c)] : \text{int}} \quad \textsc{Const}\ \dfrac{}{\Gamma \vdash 2 : \text{int}\ \checkmark}}{\Gamma \vdash *a[f(b \to c)] + 2 : \text{int}}$$

but what do we do with $\leq$?

Equality of Types =

Summary of Type Checking

- Choosing which rule to apply at an AST node is determined by the type of the child nodes
- determining the rule requires a check for $\leadsto$ *equality* of types

type equality in C:

- **`struct`** `A {}` and **`struct`** `B {}` are considered to be different
  - $\leadsto$ the compiler could re-order the fields of `A` and `B` independently (*not* allowed in C)
  - to extend an record `A` with more fields, it has to be embedded into another record:

```
struct B {
    struct A;
    int field_of_B;
} extension_of_A;
```

- after issuing **`typedef int`** `C;` the types `C` and **`int`** are the same

Structural Type Equality

Alternative interpretation of type equality (*does not hold in C*):

semantically, two types t_1, t_2 can be considered as *equal* if they accept the same set of access paths.

Example:

```
struct list {
  int info;
  struct list* next;
}
```

```
struct list1 {
  int info;
  struct {
    int info;
    struct list1* next;
  }* next;
}
```

Consider declarations `struct list* l` and `struct list1* l`. Both allow

`l->info` `l->next->info`

but the two declarations of `l` have unequal types in C.

Algorithm for Testing Structural Equality

Idea:

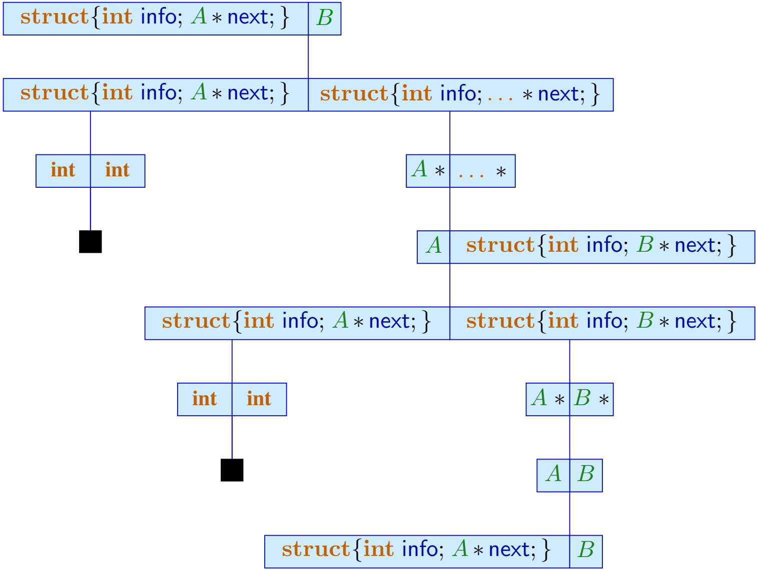
- track a set of equivalence queries of type expressions
- if two types are syntactically equal, we stop and report success
- otherwise, reduce the equivalence query to a several equivalence queries on (hopefully) simpler type expressions

Suppose that recursive types were introduced using type definitions:

$$\texttt{typedef}\ A\ t$$

(we omit the Γ). Then define the following rules:

Rules for Well-Typedness



Example:

$$\begin{array}{ll} \textbf{typedef} \quad \textbf{struct } \{\textbf{int } \mathsf{info};\ A * \mathsf{next};\} & A \\ \textbf{typedef} \quad \textbf{struct } \{\textbf{int } \mathsf{info};\ \textbf{struct } \{\textbf{int } \mathsf{info};\ B * \mathsf{next};\} * \mathsf{next};\} & B \end{array}$$

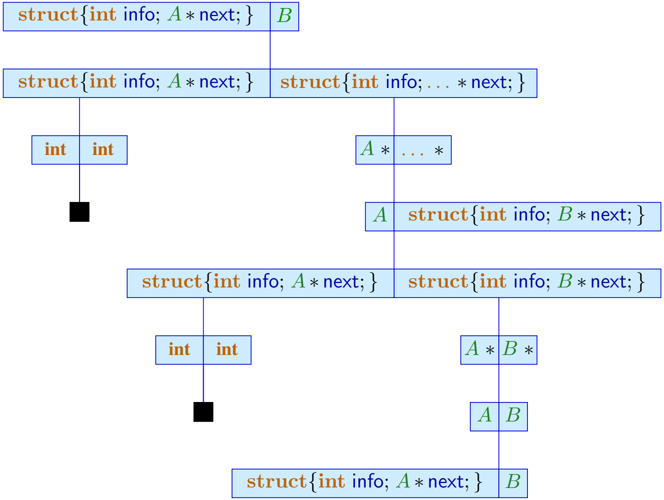
We ask, for instance, if the following equality holds:

$$\textbf{struct } \{\textbf{int } \mathsf{info};\ A * \mathsf{next};\} = B$$

We construct the following deduction tree:

Proof for the Example:

$$\textbf{typedef} \quad \textbf{struct}\ \{\textbf{int}\ \mathsf{info};\ A * \mathsf{next};\} \qquad A$$
$$\textbf{typedef} \quad \textbf{struct}\ \{\textbf{int}\ \mathsf{info};\ \textbf{struct}\ \{\textbf{int}\ \mathsf{info};\ B * \mathsf{next};\} * \mathsf{next};\} \qquad B$$



Implementation

We implement a function that implements the equivalence query for two types by applying the deduction rules:

- if no deduction rule applies, then the two types are *not equal*
- if the deduction rule for expanding a type definition applies, the function is called recursively with a *potentially larger* type
- in case an equivalence query appears a second time, the types are *equal by definition*

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Termination

- the set D of all declared types is finite
- there are no more than $|D|^2$ different equivalence queries
- repeated queries for the same inputs are automatically satisfied

$\leadsto$ termination is ensured

Subtyping $\leq$

On the arithmetic basic types **`char`**, **`int`**, **`long`**, etc. there exists a rich *subtype* hierarchy

Subtypes

$t_1 \leq t_2$, means that the values of type t_1

1. form a subset of the values of type t_2;
2. can be converted into a value of type t_2;
3. fulfill the requirements of type t_2;
4. are assignable to variables of type $t2$.

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Example:
assign smaller type (fewer values) to larger type (more values)

$$\begin{array}{l} t_1 \quad x; \\ t_2 \quad y; \\ y = x; \end{array}$$

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t_2 \quad y; \\
y = x; \\
t_1 \leq t_2
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$$
\begin{array}{l}
\texttt{int}\ x; \\
\texttt{double}\ y; \\
y = x; \\
\texttt{int} \leq \texttt{double}
\end{array}
$$

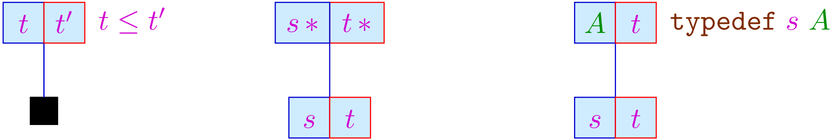
Example: Subtyping

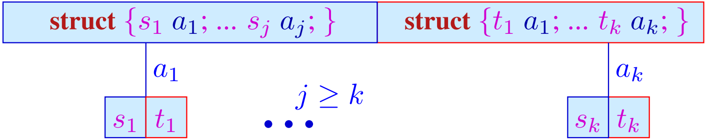
Extending the subtype relationship to more complex types, observe:

```
string extractInfo( struct { string info; } x) {
  return x.info;
}
```

- we want `extractInfo` to be applicable to all argument structures that return a `string` typed field for accessor `info`
- the idea of subtyping on values is related to subclasses
- we use deduction rules to describe when $t_1 \leq t_2$ should hold...

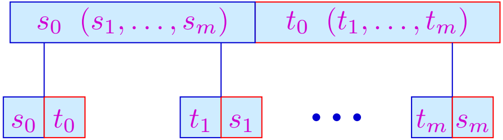
Rules for Well-Typedness of Subtyping





struct $\{int\ u, int\ v\}$ $x;$
struct $\{int\ u\}$ $y;$
$y = x;$

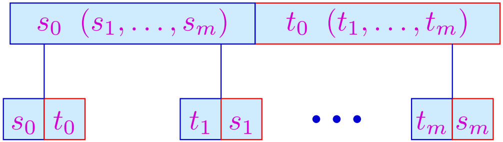
Rules and Examples for Subtyping



Examples:

struct $\{$**int** a; **int** b; $\}$	**struct** $\{$**float** a; $\}$
int (**int**)	**float** (**float**)
int (**float**)	**float** (**int**)

Rules and Examples for Subtyping



Examples:

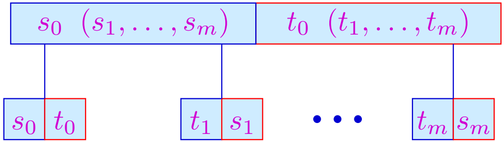
struct $\{$**int** a; **int** b; $\}$	**struct** $\{$**float** a; $\}$
int (**int**)	**float** (**float**)
int (**float**)	**float** (**int**)

Definition

Given two function types in subtype relation $s_0(s_1, \ldots s_n) \leq t_0(t_1, \ldots t_n)$ then we have

- co-variance of the return type $s_0 \leq t_0$ and
- contra-variance of the arguments $s_i \geq t_i$ für $1 < i \leq n$

Rules and Examples for Subtyping



Examples:

$$
\begin{array}{lcl}
\textbf{struct}\ \{\textbf{int}\ a;\ \textbf{int}\ b;\} & \leq & \textbf{struct}\ \{\textbf{float}\ a;\} \\
\textbf{int}\ (\textbf{int}) & \not\leq & \textbf{float}\ (\textbf{float}) \\
\textbf{int}\ (\textbf{float}) & \leq & \textbf{float}\ (\textbf{int})
\end{array}
$$

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- co-variance of the return type $s_0 \leq t_0$ and
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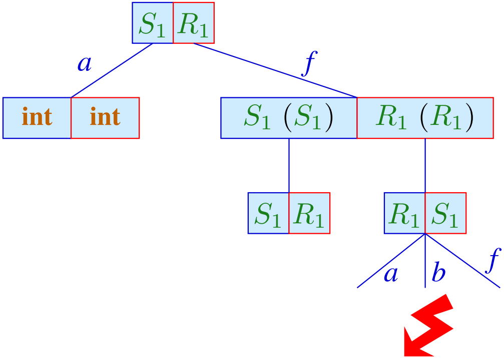
Subtypes: Application of Rules (I)

Check if $S_1 \leq R_1$:

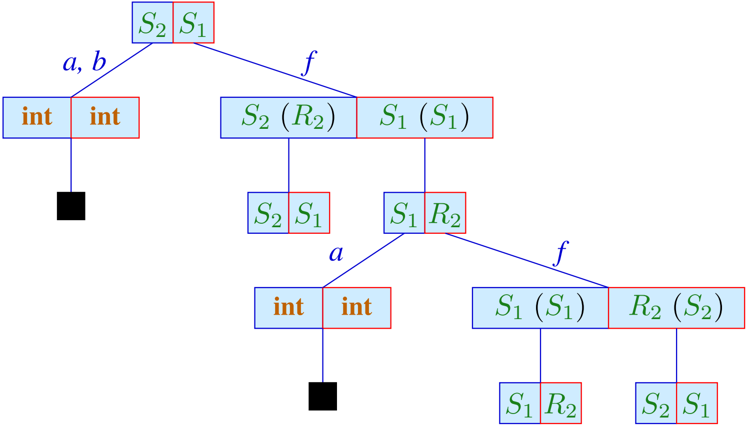
$$\begin{array}{lll} R_1 & = & \textbf{struct}\ \{\textbf{int}\ a;\ R_1\ (R_1)\ f;\} \\ S_1 & = & \textbf{struct}\ \{\textbf{int}\ a;\ \textbf{int}\ b;\ S_1\ (S_1)\ f;\} \\ R_2 & = & \textbf{struct}\ \{\textbf{int}\ a;\ R_2\ (S_2)\ f;\} \\ S_2 & = & \textbf{struct}\ \{\textbf{int}\ a;\ \textbf{int}\ b;\ S_2\ (R_2)\ f;\} \end{array}$$



Subtypes: Application of Rules (II)

Check if $S_2 \leq S_1$:

$$\begin{array}{lcl} R_1 & = & \textbf{struct}\ \{\textbf{int}\ a;\ R_1\ (R_1)\ f;\} \\ S_1 & = & \textbf{struct}\ \{\textbf{int}\ a;\ \textbf{int}\ b;\ S_1\ (S_1)\ f;\} \\ R_2 & = & \textbf{struct}\ \{\textbf{int}\ a;\ R_2\ (S_2)\ f;\} \\ S_2 & = & \textbf{struct}\ \{\textbf{int}\ a;\ \textbf{int}\ b;\ S_2\ (R_2)\ f;\} \end{array}$$



Subtypes: Application of Rules (III)

Check if $S_2 \leq R_1$:

$$
\begin{array}{lcl}
R_1 & = & \textbf{struct}\ \{\textbf{int}\ a;\ R_1\,(R_1)\ f;\} \\
S_1 & = & \textbf{struct}\ \{\textbf{int}\ a;\ \textbf{int}\ b;\ S_1\,(S_1)\ f;\} \\
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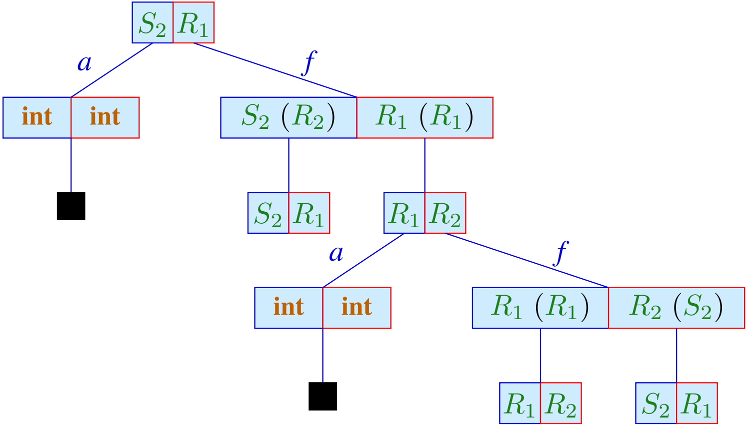
\end{array}
$$



Discussion

- for presentational purposes, proof trees are often abbreviated by omitting deductions within the tree
- structural sub-types are very powerful and can be quite intricate to understand
- Java generalizes structs to objects/classes where a sub-class A inheriting form base class O is a subtype $A \leq O$
- subtype relations between classes must be explicitly declared