Topic:

Syntactic Analysis

Syntactic Analysis

Token-Stream → Parser → Syntaxtree

- Syntactic analysis tries to integrate Tokens into larger program units.
- Such units may possibly be:
  - → Expressions;
  - → Statements;
  - → Conditional branches;
  - → loops; ...

Discussion:

In general, parsers are not developed by hand, but generated from a specification:

Specification → Generator → Parser   E→E{op}E → Ge

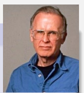
Specification of the hierarchical structure: contextfree grammars
Generated implementation: Pushdown automata + X

Chapter 1:
Basics of Contextfree Grammars

Basics: Context-free Grammars

- Programs of programming languages can have arbitrary numbers of tokens, but only finitely many Token-classes.
- This is why we choose the set of Token-classes to be the finite alphabet of terminals T.
- The nested structure of program components can be described elegantly via context-free grammars...


Noam Chomsky


John Backus

Definition: Context-Free Grammar

A context-free grammar (CFG) is a 4-tuple $G = (N, T, P, S)$ with:

- N the set of nonterminals,
- T the set of terminals,
- P the set of productions or rules, and
- $S \in N$ the start symbol

Conventions

The rules of context-free grammars take the following form:

$$A \to \alpha \quad \text{with} \quad A \in N\,,\ \ \alpha \in (N \cup T)^*$$

... for example:

$$\begin{array}{lcl} S & \to & a\,S\,b \\ S & \to & \epsilon \end{array}$$

Specified language: $\{a^n b^n \mid n \geq 0\}$

Conventions:

In examples, we specify nonterminals and terminals in general implicitely:

- nonterminals are: $A, B, C, \ldots$, ⟨exp⟩, ⟨stmt⟩, ...;
- terminals are: $a, b, c, \ldots$, int, name, ...;

... a practical example:

$$\begin{array}{lcl}
S & \to & \langle\text{stmt}\rangle \\
\langle\text{stmt}\rangle & \to & \langle\text{if}\rangle \mid \langle\text{while}\rangle \mid \langle\text{rexp}\rangle; \\
\langle\text{if}\rangle & \to & \textbf{if}\ (\ \langle\text{rexp}\rangle\ )\ \langle\text{stmt}\rangle\ \textbf{else}\ \langle\text{stmt}\rangle \\
\langle\text{while}\rangle & \to & \textbf{while}\ (\ \langle\text{rexp}\rangle\ )\ \langle\text{stmt}\rangle \\
\langle\text{rexp}\rangle & \to & \textbf{int} \mid \langle\text{lexp}\rangle \mid \langle\text{lexp}\rangle = \langle\text{rexp}\rangle \mid \ldots \\
\langle\text{lexp}\rangle & \to & \textbf{name} \mid \ldots
\end{array}$$

More conventions:

- For every nonterminal, we collect the right hand sides of rules and list them together.
- The j-th rule for A can be identified via the pair (A, j) (with $j \geq 0$).

Pair of grammars:

$$\begin{array}{lcl}
E & \to & E{+}E \mid E{*}E \mid (\ E\ ) \mid \textbf{name} \mid \textbf{int}
\end{array}$$

$$\begin{array}{lcl}
E & \to & E{+}T \mid T \\
T & \to & T{*}F \mid F \\
F & \to & (\ E\ ) \mid \textbf{name} \mid \textbf{int}
\end{array}$$

$$\begin{array}{lcl}
E & \to & E{+}E\ ^0 \mid E{*}E\ ^1 \mid (\ E\ )\ ^2 \mid \textbf{name}\ ^3 \mid \textbf{int}\ ^4
\end{array}$$

$$\begin{array}{lcl}
E & \to & E{+}T\ ^0 \mid T\ ^1 \\
T & \to & T{*}F\ ^0 \mid F\ ^1 \\
F & \to & (\ E\ )\ ^0 \mid \textbf{name}\ ^1 \mid \textbf{int}\ ^2
\end{array}$$

Both grammars describe the same language

Derivation

Grammars are term rewriting systems. The rules offer feasible rewriting steps. A sequence of such rewriting steps $\alpha_0 \to \ldots \to \alpha_m$ is called derivation.

... for example:

$$\begin{array}{rl}
\underline{E} & \to\ \underline{E} + T \\
& \to\ \underline{T} + T \\
& \to\ T * \underline{F} + T \\
& \to\ \underline{T} * \textbf{int} + T \\
& \to\ \underline{F} * \textbf{int} + T \\
& \to\ \textbf{name} * \textbf{int} + \underline{T} \\
& \to\ \textbf{name} * \textbf{int} + \underline{F} \\
& \to\ \textbf{name} * \textbf{int} + \textbf{int}
\end{array}$$

Definition

The rewriting relation $\to$ is a relation on words over $N \cup T$, with

$$\alpha \to \alpha' \quad \text{iff} \quad \alpha = \alpha_1\ A\ \alpha_2 \ \wedge\ \alpha' = \alpha_1\ \beta\ \alpha_2 \quad \text{for an} \quad A \to \beta \in P$$

The reflexive and transitive closure of $\to$ is denoted as: $\to^*$

Derivation

Remarks:

- The relation $\to$ depends on the grammar
- In each step of a derivation, we may choose:
  - * a spot, determining where we will rewrite.
  - * a rule, determining how we will rewrite.
- The language, specified by G is:

$$\mathcal{L}(G) = \{w \in T^* \mid S \to^* w\}$$

Attention:

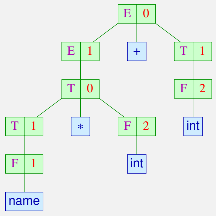
The order, in which disjunct fragments are rewritten is not relevant.

Derivation Tree

Derivations of a symbol are represented as derivation trees:

... for example:

$$
\begin{array}{ll}
\underline{E} & \to^0 \ \underline{E} + T \\
& \to^1 \ \underline{T} + T \\
& \to^0 \ T * \underline{F} + T \\
& \to^2 \ \underline{T} * \text{int} + T \\
& \to^1 \ \underline{F} * \text{int} + T \\
& \to^1 \ \text{name} * \text{int} + \underline{T} \\
& \to^1 \ \text{name} * \text{int} + \underline{F} \\
& \to^2 \ \text{name} * \text{int} + \text{int}
\end{array}
$$



A derivation tree for $A \in N$:
inner nodes: rule applications
root: rule application for A
leaves: terminals or ϵ
The successors of (B, i) correspond to right hand sides of the rule

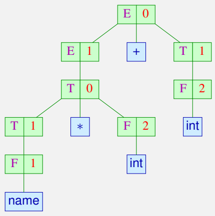
Special Derivations

Attention:
In contrast to arbitrary derivations, we find special ones, always rewriting the leftmost (or rather rightmost) occurance of a nonterminal.

- These are called leftmost (or rather rightmost) derivations and are denoted with the index L (or R respectively).
- Leftmost (or rightmost) derivations correspondt to a left-to-right (or right-to-left) preorder-DFS-traversal of the derivation tree.
- Reverse rightmost derivations correspond to a left-to-right postorder-DFS-traversal of the derivation tree

Special Derivations

... for example:

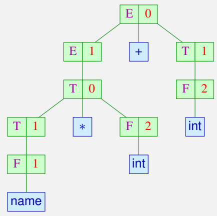


Leftmost derivation: $(E,0)\,(E,1)\,(T,0)\,(T,1)\,(F,1)\,(F,2)\,(T,1)\,(F,2)$
Rightmost derivation: $(E,0)\,(T,1)\,(F,2)\,(E,1)\,(T,0)\,(F,2)\,(T,1)\,(F,1)$
Reverse rightmost derivation: $(F,1)\,(T,1)\,(F,2)\,(T,0)\,(E,1)\,(F,2)\,(T,1)\,(E,0)$

Unique Grammars

The concatenation of leaves of a derivation tree t are often called $\text{yield}(t)$.

... for example:



gives rise to the concatenation: $\text{name} * \text{int} + \text{int}$.

Unique Grammars

Definition:
Grammar G is called unique, if for every $w \in T^*$ there is maximally one derivation tree t of S with $\text{yield}(t) = w$.

... in our example:

$$
E \to E{+}E^{\,0} \mid E{*}E^{\,1} \mid (\,E\,)^{\,2} \mid \text{name}^{\,3} \mid \text{int}^{\,4}
$$

$$
\begin{array}{lll}
E & \to & E{+}T^{\,0} \mid T^{\,1} \\
T & \to & T{*}F^{\,0} \mid F^{\,1} \\
F & \to & (\,E\,)^{\,0} \mid \text{name}^{\,1} \mid \text{int}^{\,2}
\end{array}
$$

The first one is ambiguous, the second one is unique

Conclusion:

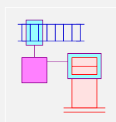
- A derivation tree represents a possible hierarchical structure of a word.
- For programming languages, only those grammars with a unique structure are of interest.
- Derivation trees are one-to-one corresponding with leftmost derivations as well as (reverse) rightmost derivations.

- Leftmost derivations correspond to a top-down reconstruction of the syntax tree.
- Reverse rightmost derivations correspond to a bottom-up reconstruction of the syntax tree.

Chapter 2:
Basics of Pushdown Automata

Basics of Pushdown Automata

Languages, specified by context free grammars are accepted by Pushdown Automata:



The pushdown is used e.g. to verify correct nesting of braces.

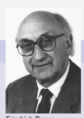
Example:

States: $0, 1, 2$
Start state: 0
Final states: $0, 2$

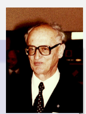
0	a	11
1	a	11
11	b	2
12	b	2

Conventions:
- We do not differentiate between pushdown symbols and states
- The rightmost / upper pushdown symbol represents the state
- Every transition consumes / modifies the upper part of the pushdown

Definition: Pushdown Automaton



Friedrich Bauer



Klaus Samelson

A pushdown automaton (PDA) is a tuple $M = (Q, T, \delta, q_0, F)$ with:
- Q a finite set of states;
- T an input alphabet;
- $q_0 \in Q$ the start state;
- $F \subseteq Q$ the set of final states and
- $\delta \subseteq Q^+ \times (T \cup \{\epsilon\}) \times Q^*$ a finite set of transitions

We define computations of pushdown automata with the help of transitions; a particular computation state (the current configuration) is a pair:

$$(\gamma, w) \ \in \ Q^* \times T^*$$

consisting of the pushdown content and the remaining input.

... for example:

States: $0, 1, 2$
Start state: 0
Final states: $0, 2$

0	a	11
1	a	11
11	b	2
12	b	2

$$\begin{array}{lll} (0\,, & a\,a\,a\,b\,b\,b) & \vdash & (1\,1\,, & a\,a\,b\,b\,b) \\ & & \vdash & (1\,1\,1\,, & a\,b\,b\,b) \\ & & \vdash & (1\,1\,1\,1\,, & b\,b\,b) \\ & & \vdash & (1\,1\,2\,, & b\,b) \\ & & \vdash & (1\,2\,, & b) \\ & & \vdash & (2\,, & \epsilon) \end{array}$$

A computation step is characterized by the relation $\vdash \subseteq (Q^* \times T^*)^2$ with

$$(\alpha\,\gamma,\, x\,w) \vdash (\alpha\,\gamma',\, w) \quad \text{for} \quad (\gamma,\, x,\, \gamma') \in \delta$$

Remarks:

- The relation $\vdash$ depends on the pushdown automaton M
- The reflexive and transitive closure of $\vdash$ is denoted by $\vdash^*$
- Then, the language accepted by M is

$$\mathcal{L}(M) = \{w \in T^* \mid \exists\, f \in F : (q_0, w) \vdash^* (f, \epsilon)\}$$

We accept with a final state together with empty input.

Definition: Deterministic Pushdown Automaton
The pushdown automaton M is deterministic, if every configuration has maximally one successor configuration.

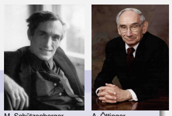
This is exactly the case if for distinct transitions $(\gamma_1, x, \gamma_2)\,,\ (\gamma_1', x', \gamma_2') \in \delta$ we can assume:
Is γ_1 a suffix of γ_1', then $x \neq x' \wedge x \neq \epsilon \neq x'$ is valid.

... for example:

0	a	11
1	a	11
11	b	2
12	b	2

... this obviously holds

Pushdown Automata



M. Schützenberger   A. Öttinger

Theorem:
For each context free grammar $G = (N, T, P, S)$
a pushdown automaton M with $\mathcal{L}(G) = \mathcal{L}(M)$ can be built.

The theorem is so important for us, that we take a look at two constructions for automata, motivated by both of the special derivations:

- M_G^L to build Leftmost derivations
- M_G^R to build reverse Rightmost derivations

Chapter 3:
Top-down Parsing

Item Pushdown Automaton

Construction: Item Pushdown Automaton M_G^L

- Reconstruct a Leftmost derivation.
- Expand nonterminals using a rule.
- Verify successively, that the chosen rule matches the input.

$\Longrightarrow$ The states are now Items (= rules with a bullet):

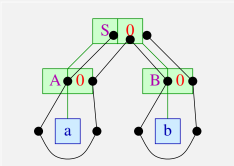
$$[A \to \alpha \bullet \beta]\,, \qquad A \to \alpha\,\beta \in P$$

The bullet marks the spot, how far the rule is already processed

Item Pushdown Automaton – Example

Our example:

$$S \to A\,B^0 \qquad A \to a^0 \qquad B \to b^0$$



Item Pushdown Automaton – Example

We add another rule $S' \to S\,\$$ for initialising the construction:

Start state: $[S' \to \bullet\ S\,\$]$
End state: $[S' \to S\ \bullet\ \$]$
Transition relations:

$[S' \to \bullet\ S\,\$]$	ϵ	$[S' \to \bullet\ S\,\$]\,[S \to \bullet\ A\,B]$
$[S \to \bullet\ A\,B]$	ϵ	$[S \to \bullet\ A\,B]\,[A \to \bullet\ a]$
$[A \to \bullet\, a]$	a	$[A \to a\,\bullet]$
$[S \to \bullet\ A\,B]\,[A \to a\,\bullet]$	ϵ	$[S \to A\ \bullet\ B]$
$[S \to A\ \bullet\ B]$	ϵ	$[S \to A\ \bullet\ B]\,[B \to \bullet\ b]$
$[B \to \bullet\, b]$	b	$[B \to b\,\bullet]$
$[S \to A\ \bullet\ B]\,[B \to b\,\bullet]$	ϵ	$[S \to A\,B\,\bullet]$
$[S' \to \bullet\ S\,\$]\,[S \to A\,B\,\bullet]$	ϵ	$[S' \to S\ \bullet\ \$]$

Item Pushdown Automaton

The item pushdown automaton M_G^L has three kinds of transitions:

Expansions: $([A \to \alpha \bullet B\,\beta], \epsilon, [A \to \alpha \bullet B\,\beta]\,[B \to \bullet\,\gamma])$ for $A \to \alpha\,B\,\beta,\ B \to \gamma \in P$
Shifts: $([A \to \alpha \bullet a\,\beta], a, [A \to \alpha\,a \bullet \beta])$ for $A \to \alpha\,a\,\beta \in P$
Reduces: $([A \to \alpha \bullet B\,\beta]\,[B \to \gamma\bullet], \epsilon, [A \to \alpha\,B \bullet \beta])$ for $A \to \alpha\,B\,\beta,\ B \to \gamma \in P$

Items of the form: $[A \to \alpha\,\bullet]$ are also called complete
The item pushdown automaton shifts the bullet around the derivation tree ...

Item Pushdown Automaton

Discussion:

- The expansions of a computation form a leftmost derivation
- Unfortunately, the expansions are chosen nondeterministically
- For proving correctness of the construction, we show that for every Item $[A \to \alpha \bullet B\,\beta]$ the following holds:

$$([A \to \alpha \bullet B\,\beta], w) \vdash^* ([A \to \alpha\,B \bullet \beta], \epsilon) \quad \text{iff} \quad B \to^* w$$

- LL-Parsing is based on the item pushdown automaton and tries to make the expansions deterministic ...

Item Pushdown Automaton

Example: $S' \to S\,\$ \qquad S \to \epsilon \mid a\,S\,b$

The transitions of the according Item Pushdown Automaton:

0	$[S' \to \bullet\,S\,\$]$	ϵ	$[S' \to \bullet\,S\,\$]\,[S \to \bullet]$
1	$[S' \to \bullet\,S\,\$]$	ϵ	$[S' \to \bullet\,S\,\$]\,[S \to \bullet\,a\,S\,b]$
2	$[S \to \bullet\,a\,S\,b]$	a	$[S \to a \bullet S\,b]$
3	$[S \to a \bullet S\,b]$	ϵ	$[S \to a \bullet S\,b]\,[S \to \bullet]$
4	$[S \to a \bullet S\,b]$	ϵ	$[S \to a \bullet S\,b]\,[S \to \bullet\,a\,S\,b]$
5	$[S \to a \bullet S\,b]\,[S \to \bullet]$	ϵ	$[S \to a\,S \bullet b]$
6	$[S \to a \bullet S\,b]\,[S \to a\,S\,b\bullet]$	ϵ	$[S \to a\,S \bullet b]$
7	$[S \to a\,S \bullet b]$	b	$[S \to a\,S\,b\bullet]$
8	$[S' \to \bullet\,S\,\$]\,[S \to \bullet]$	ϵ	$[S' \to S \bullet \$]$
9	$[S' \to \bullet\,S\,\$]\,[S \to a\,S\,b\bullet]$	ϵ	$[S' \to S \bullet \$]$

Conflicts arise between the transitions $(0, 1)$ and $(3, 4)$, resp..

Topdown Parsing

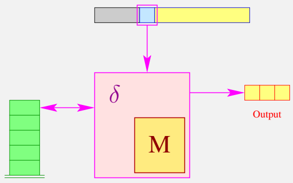
Problem:
Conflicts between the transitions prohibit an implementation of the item pushdown automaton as deterministic pushdown automaton.

Idea 1: GLL Parsing
For each conflict, we create a virtual copy of the complete configuration and continue computing in parallel.

Idea 2: Recursive Descent & Backtracking
Depth-first search for an appropriate derivation.

Idea 3: Recursive Descent & Lookahead
Conflicts are resolved by considering a lookup of the next input symbols.

Structure of the $LL(1)$-Parser:



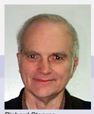
- The parser accesses a frame of length 1 of the input;
- it corresponds to an item pushdown automaton, essentially;
- table $M[q, w]$ contains the rule of choice.

Topdown Parsing

Idea:

- Emanate from the item pushdown automaton
- Consider the next input symbol to determine the appropriate rule for the next expansion
- A grammar is called $LL(1)$ if a unique choice is always possible


Philip Lewis


Richard Stearns

Definition:
A reduced grammar is called $LL(1)$, if for each two distinct rules $A \to \alpha\,,\ \ A \to \alpha' \ \in\ P$ and each derivation $S \to^*_L u\,A\,\beta$ with $u \in T^*$ the following is valid:

$$\text{First}_1(\alpha\,\beta)\ \cap\ \text{First}_1(\alpha'\,\beta) = \emptyset$$

Topdown Parsing

Example 1:

$$\begin{array}{lcl} S & \to & \textbf{if}\ (\ E\ )\ S\ \textbf{else}\ S \quad | \\ & & \textbf{while}\ (\ E\ )\ S \quad | \\ & & E\ ; \\ E & \to & \textbf{id} \end{array}$$

is $LL(1)$, since $\quad \text{First}_1(E) = \{\textbf{id}\}$

Example 2:

$$\begin{array}{lcl} S & \to & \textbf{if}\ (\ E\ )\ S\ \textbf{else}\ S \quad | \\ & & \textbf{if}\ (\ E\ )\ S \quad | \\ & & \textbf{while}\ (\ E\ )\ S \quad | \\ & & E\ ; \\ E & \to & \textbf{id} \end{array}$$

... is not $LL(k)$ for any $k > 0$.

Lookahead Sets

Definition: First_1-Sets
For a set $L \subseteq T^*$ we define:

$$\text{First}_1(L)\ =\ \{\epsilon \mid \epsilon \in L\} \cup \{u \in T \mid \exists\, v \in T^*\ :\ \ uv \in L\}$$

Example: $S \to \epsilon \quad | \quad a\,S\,b$

$\text{First}_1(⟦S⟧)$
ϵ
$a\,b$
$a\,a\,b\,b$
$a\,a\,a\,b\,b\,b$
$\cdots$

$\equiv$ the yield's prefix of length 1

Lookahead Sets

Arithmetics:
$\text{First}_1(\_)$ is distributive with union and concatenation:

$$\begin{array}{lcl} \text{First}_1(\emptyset) & = & \emptyset \\ \text{First}_1(L_1\ \cup\ L_2) & = & \text{First}_1(L_1)\ \cup\ \text{First}_1(L_2) \\ \text{First}_1(L_1\ \cdot\ L_2) & = & \text{First}_1(\text{First}_1(L_1)\ \cdot\ \text{First}_1(L_2)) \\ & := & \text{First}_1(L_1)\ \odot_1\ \text{First}_1(L_2) \end{array}$$

$\odot_1$ being $1-$ concatenation

Definition: 1-concatenation
Let $L_1, L_2\ \subseteq T \cup \{\epsilon\}$ with $L_1\ \neq\ \emptyset\ \neq\ L_2$. Then:

$$L_1\ \odot_1\ L_2\ =\ \begin{cases} L_1 & \text{if}\quad \epsilon \not\in L_1 \\ (L_1 \backslash \{\epsilon\}) \cup L_2 & \text{otherwise} \end{cases}$$

If all rules of G are productive, then all sets $\text{First}_1(A)$ are non-empty.

Lookahead Sets

For $\alpha \in (N \cup T)^*$ we are interested in the set:

$$\text{First}_1(\alpha)\ =\ \text{First}_1(\{w \in T^* \mid \alpha \to^* w\})$$

Idea: Treat ϵ separately: $\text{First}_1(A) = F_\epsilon(A) \cup \{\epsilon \mid A \to^* \epsilon\}$

- Let $\quad \text{empty}(X) = \text{true} \quad$ iff $\quad X \to^* \epsilon$.
- $F_\epsilon(X_1 \ldots X_m) = \bigcup_{i=1}^{j} F_\epsilon(X_i)$ if $\neg\text{empty}(X_j)\ \wedge\ \bigwedge_{i=1}^{j-1} \text{empty}(X_i)$

We characterize the ϵ-free First_1-sets with an inequality system:

$$\begin{array}{lcll} F_\epsilon(a) & = & \{a\} & \text{if} \quad a \in T \\ F_\epsilon(A) & \supseteq & F_\epsilon(X_j) & \text{if} \quad A \to X_1 \ldots X_m\ \in P, \quad \text{empty}(X_1) \wedge \ldots \wedge \text{empty}(X_{j-1}) \end{array}$$

Lookahead Sets

for example...

$$\begin{array}{lcl} E & \to & E + T \quad | \quad T \\ T & \to & T * F \quad | \quad F \\ F & \to & (\ E\ ) \quad | \quad \textbf{name} \quad | \quad \textbf{int} \end{array}$$

with $\quad \text{empty}(E) = \text{empty}(T) = \text{empty}(F) = \text{false}$

... we obtain:

$$\begin{array}{lcl@{\qquad}lcl} F_\epsilon(S') & \supseteq & F_\epsilon(E) & F_\epsilon(E) & \supseteq & F_\epsilon(E) \\ F_\epsilon(E) & \supseteq & F_\epsilon(T) & F_\epsilon(T) & \supseteq & F_\epsilon(T) \\ F_\epsilon(T) & \supseteq & F_\epsilon(F) & F_\epsilon(F) & \supseteq & \{\,(\,, \textbf{name}, \textbf{int}\} \end{array}$$

Fast Computation of Lookahead Sets

Observation:

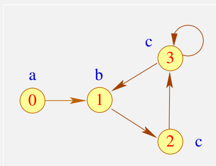
- The form of each inequality of these systems is:

  $$x\ \sqsupseteq\ y \qquad \text{resp.} \qquad x\ \sqsupseteq\ d$$

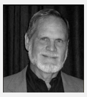
  for variables x, y und $d\ \in\ \mathbb{D}$.
- Such systems are called pure unification problems
- Such problems can be solved in linear space/time.

for example: $\quad \mathbb{D} = 2^{\{a,b,c\}}$

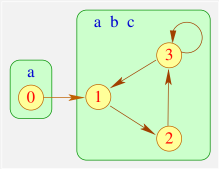
$$\begin{array}{lll} x_0 \supseteq \{a\} & & \\ x_1 \supseteq \{b\} & x_1 \supseteq x_0 & x_1 \supseteq x_3 \\ x_2 \supseteq \{c\} & x_2 \supseteq x_1 & \\ x_3 \supseteq \{c\} & x_3 \supseteq x_2 & x_3 \supseteq x_3 \end{array}$$



Fast Computation of Lookahead Sets



Frank DeRemer & Tom Pennello



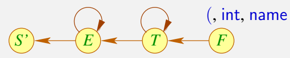
Proceeding:

- Create the Variable Dependency Graph for the inequality system.
- Whithin a Strongly Connected Component (→ Tarjan) all variables have the same value
- Is there no ingoing edge for an SCC, its value is computed via the smallest upper bound of all values within the SCC
- In case of ingoing edges, their values are also to be considered for the upper bound

Fast Computation of Lookahead Sets

... for our example grammar:

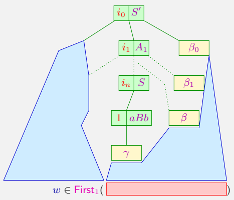
First_1 :

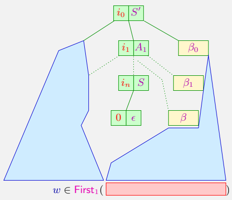


Item Pushdown Automaton as LL(1)-Parser

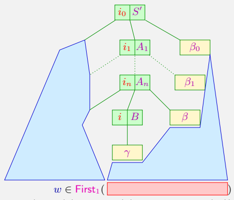
context is relevant too: $S' \to S\,\$ \qquad S \to \epsilon^{\,0} \;\mid\; a\,S\,b^{\,1}$

$\text{First}_1(\text{input})$	$\$$	a	b
S	?	?	?





Item Pushdown Automaton as LL(1)-Parser



$w \in \text{First}_1(\quad)$
$w \in \text{First}_1(\text{First}_1(\gamma) \odot_1 \text{First}_1(\beta) \odot_1 \ldots \odot_1 \text{First}_1(\beta_0))$
$w \in \text{First}_1(\gamma) \odot_1 \text{Follow}_1(B)$

Inequality system for $\text{Follow}_1(B) = \text{First}_1(\beta) \odot_1 \ldots \odot_1 \text{First}_1(\beta_0)$

$$\begin{array}{lll}
\text{Follow}_1(S) & \supseteq & \{\$\} \\
\text{Follow}_1(B) & \supseteq & F_\epsilon(X_j) \quad \text{if} \quad A \to \alpha\, B\, X_1 \ldots X_m \;\in P,\ \text{empty}(X_1) \wedge \ldots \wedge \text{empty}(X_{j-1}) \\
\text{Follow}_1(B) & \supseteq & \text{Follow}_1(A) \quad \text{if} \quad A \to \alpha\, B\, X_1 \ldots X_m \;\in P,\ \text{empty}(X_1) \wedge \ldots \wedge \text{empty}(X_m)
\end{array}$$

Item Pushdown Automaton as LL(1)-Parser

Is G an $LL(1)$-grammar, we can index a lookahead-table with items and nonterminals:

LL(1)-Lookahead Table

We set $M[B,\ w] \;=\; i$ with $B \to \gamma^{\,i}$ if $w \in \text{First}_1(\gamma) \odot_1 \text{Follow}_1(B)$

... for example: $S' \to S\,\$ \qquad S \to \epsilon^{\,0} \;\mid\; a\,S\,b^{\,1}$

$\text{First}_1(S) = \{\epsilon, a\} \quad \text{Follow}_1(S) = \{b, \$\}$

S-rule 0 : $\text{First}_1(\epsilon) \;\odot_1\; \text{Follow}_1(S) = \{b, \$\}$
S-rule 1 : $\text{First}_1(aSb) \;\odot_1\; \text{Follow}_1(S) = \{a\}$

	$\$$	a	b
S	0	1	0

Item Pushdown Automaton as LL(1)-Parser

For example: $S' \to S\,\$ \qquad S \to \epsilon^{\,0} \mid a\,S\,b^{\,1}$
The transitions of the according Item Pushdown Automaton:

0	$[S' \to \bullet\, S\, \$]$	ϵ	$[S' \to \bullet\, S\, \$]\, [S \to \bullet]$
1	$[S' \to \bullet\, S\, \$]$	ϵ	$[S' \to \bullet\, S\, \$]\, [S \to \bullet\, a\, S\, b]$
2	$[S \to \bullet\, a\, S\, b]$	a	$[S \to a \bullet S\, b]$
3	$[S \to a \bullet S\, b]$	ϵ	$[S \to a \bullet S\, b]\, [S \to \bullet]$
4	$[S \to a \bullet S\, b]$	ϵ	$[S \to a \bullet S\, b]\, [S \to \bullet\, a\, S\, b]$
5	$[S \to a \bullet S\, b]\, [S \to \bullet]$	ϵ	$[S \to a\, S \bullet b]$
6	$[S \to a \bullet S\, b]\, [S \to a\, S\, b \bullet]$	ϵ	$[S \to a\, S \bullet b]$
7	$[S \to a\, S \bullet b]$	b	$[S \to a\, S\, b \bullet]$
8	$[S' \to \bullet\, S\, \$]\, [S \to \bullet]$	ϵ	$[S' \to S \bullet \$]$
9	$[S' \to \bullet\, S\, \$]\, [S \to a\, S\, b \bullet]$	ϵ	$[S' \to S \bullet \$]$

Lookahead table:

	$\$$	a	b
S	0	1	0

Left Recursion

Attention:
Many grammars are not $LL(k)$!

A reason for that is:

Definition
Grammar G is called left-recursive, if

$$A \to^{+} A\,\beta \qquad \text{for an} \quad A \in N,\ \beta \in (T \cup N)^{*}$$

Example:

$$\begin{array}{lcl}
E & \to & E + T \;\mid\; T \\
T & \to & T * F \;\mid\; F \\
F & \to & (\,E\,) \;\mid\; \text{name} \;\mid\; \text{int}
\end{array}$$

... is left-recursive

Left Recursion

Theorem:
Let a grammar G be reduced and left-recursive, then G is not $LL(k)$ for any k.

Proof:
Let wlog. $A \to A\,\beta \mid \alpha \quad \in P$
and A be reachable from S

Assumption: G is $LL(k)$

$\Rightarrow \text{First}_k(\alpha\,\beta^{n}\,\gamma) \,\cap$
$\text{First}_k(\alpha\,\beta^{n+1}\,\gamma) = \emptyset$

Case 1: $\beta \to^{*} \epsilon$ — Contradiction !!!
Case 2: $\beta \to^{*} w \neq \epsilon \Longrightarrow \text{First}_k(\alpha\, w^{k}\,\gamma) \cap \text{First}_k(\alpha\, w^{k+1}\,\gamma) \neq \emptyset$

Right-Regular Context-Free Parsing

Recurring scheme in programming languages: Lists of sth...
$S \to b \;\mid\; S\,a\,b$
Alternative idea: Regular Expressions
$S \to (\,b\,a\,)^{*}\,b$

Definition: Right-Regular Context-Free Grammar

A right-regular context-free grammar (RR-CFG) is a 4-tuple $G = (N, T, P, S)$ with:

- N the set of nonterminals,
- T the set of terminals,
- P the set of rules with regular expressions of symbols as rhs,
- $S \in N$ the start symbol

Example: Arithmetic Expressions

$$\begin{array}{lcl}
S & \to & E \\
E & \to & T\,(\,+T\,)^{*} \\
T & \to & F\,(\,*F\,)^{*} \\
F & \to & (\,E\,) \mid \text{name} \mid \text{int}
\end{array}$$

Idea 1: Rewrite the rules from G to $\langle G\rangle$:

$$\begin{array}{lcll}
A & \to & \langle\alpha\rangle & \text{if} \quad A \to \alpha \in P \\
\langle\alpha\rangle & \to & \alpha & \text{if} \quad \alpha \in N \cup T \\
\langle\epsilon\rangle & \to & \epsilon & \\
\langle\alpha^{*}\rangle & \to & \epsilon \mid \langle\alpha\rangle\langle\alpha^{*}\rangle & \text{if} \quad \alpha \in \text{Regex}_{T,N} \\
\langle\alpha_1 \ldots \alpha_n\rangle & \to & \langle\alpha_1\rangle \ldots \langle\alpha_n\rangle & \text{if} \quad \alpha_i \in \text{Regex}_{T,N} \\
\langle\alpha_1 \mid \ldots \mid \alpha_n\rangle & \to & \langle\alpha_1\rangle \mid \ldots \mid \langle\alpha_n\rangle & \text{if} \quad \alpha_i \in \text{Regex}_{T,N}
\end{array}$$

...and generate the according LL(k)-Parser $M^{L}_{\langle G\rangle}$

Example: Arithmetic Expressions cont'd

$$\begin{array}{lcl}
S & \to & E \\
E & \to & T\,(\,+T\,)^{*}\langle T\,(\,+T)^{*}\rangle \\
T & \to & F\,(\,*F\,)^{*}\langle F\,(\,*F\,)^{*}\rangle \\
F & \to & (\,E\,) \mid \text{name} \mid \text{int} \\
\langle T\,(\,+T)^{*}\rangle & \to & T\,\langle(\,+T)^{*}\rangle \\
\langle(\,+T)^{*}\rangle & \to & \epsilon \mid \langle\,+T\rangle\langle(\,+T)^{*}\rangle \\
\langle\,+T\rangle & \to & +T \\
\langle F\,(\,*F\,)^{*}\rangle & \to & F\,\langle(\,*F\,)^{*}\rangle \\
\langle(\,*F\,)^{*}\rangle & \to & \epsilon \mid \langle *F\rangle\,\langle(\,*F\,)^{*}\rangle \\
\langle\,*F\rangle & \to & *F
\end{array}$$

Definition:

An $RR-CFG$ G is called $RLL(1)$,
if the corresponding CFG $\langle G\rangle$ is an $LL(1)$ grammar.

Discussion

- directly yields the table driven parser $M^L_{\langle G\rangle}$ for $RLL(1)$ grammars
- however: mapping regular expressions to recursive productions unnessessarily strains the stack
  $\rightarrow$ instead directly construct automaton in the style of Berry-Sethi

Idea 2: Recursive Descent RLL Parsers:

Recursive descent RLL(1)-parsers are an alternative to table-driven parsers; apart from the usual function `scan()`, we generate a program frame with the lookahead function `expect()` and the main parsing method `parse()`:

```
int next;
void expect(Set E){
    if ({ε, next} ∩ E = ∅){
        cerr << "Expected" << E << "found" << next;
        exit(0);
    }
    return ;
}
void parse(){
    next = scan();
    expect(First1(S)) ;
    S();
    expect({EOF}) ;
}
```


Idea 2: Recursive Descent RLL Parsers:

For each $A \rightarrow \alpha \in P$, we introduce:

$$\texttt{void A()\{} \quad generate(\alpha) \quad \texttt{\}}$$

with the meta-program $generate$ being defined by structural decomposition of α:

$$\begin{array}{lll}
generate(r_1 \ldots r_k) & = & generate(r_1) \\
 & & \texttt{expect}(\mathsf{First}_1(r_2))\ ; \\
 & & generate(r_2) \\
 & & \vdots \\
 & & \texttt{expect}(\mathsf{First}_1(r_k))\ ; \\
 & & generate(r_k) \\
generate(\epsilon) & = & ; \\
generate(a) & = & \texttt{next} = \texttt{scan}(); \\
generate(A) & = & \texttt{A}();
\end{array}$$

Idea 2: Recursive Descent RLL Parsers:

$$\begin{array}{lll}
generate(r^*) & = & \texttt{while}\ (\ \texttt{next} \in \mathsf{F}_\epsilon(r))\ \{ \\
 & & \quad generate(r) \\
 & & \} \\
generate(r_1 \mid \ldots \mid r_k) & = & \texttt{switch}(\texttt{next})\ \{ \\
 & & \quad labels(\mathsf{First}_1(r_1))\ generate(r_1)\ \texttt{break}\ ; \\
 & & \quad \vdots \\
 & & \quad labels(\mathsf{First}_1(r_k))\ generate(r_k)\ \texttt{break}\ ; \\
 & & \} \\
labels(\{\alpha_1, \ldots, \alpha_m\}) & = & label(\alpha_1):\ \ldots\ label(\alpha_m): \\
label(\alpha) & = & \texttt{case}\ \alpha \\
label(\epsilon) & = & \texttt{default}
\end{array}$$

Topdown-Parsing

Discussion

- A practical implementation of an $RLL(1)$-parser via recursive descent is a straight-forward idea
- However, only a subset of the deterministic contextfree languages can be parsed this way.
- As soon as $\mathsf{First}_1(\_)$ sets are not disjoint any more,
  - Solution 1: For many accessibly written grammars, the alternation between right hand sides happens too early. Keeping the common prefixes of right hand sides joined and introducing a new production for the actual diverging sentence forms often helps.
  - Solution 2: Introduce *ranked* grammars, and decide conflicting lookahead always in favour of the higher ranked alternative
    $\rightarrow$ relation to LL parsing not so clear any more
    $\rightarrow$ not so clear for $\_^*$ operator how to decide
  - Solution 3: Going from $LL(1)$ to $LL(k)$
    The size of the occuring sets is rapidly increasing with larger k
    Unfortunately, even $LL(k)$ parsers are not sufficient to accept all deterministic contextfree languages. (regular lookahead $\rightarrow$ $LL(*)$)
- In practical systems, this often motivates the implementation of $k = 1$ only ...