Funding Public Projects: A Case for the Nash Product Rule

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Donor Coordination

- **Annual Charity Matching Programs of companies**
  - In 2021, Microsoft employees raised $208 million for 27,000 nonprofits and schools.\(^1\)
  - Since 2011, Apple’s Employee Giving Program has donated nearly $725 million to 39,000 organizations.\(^2\)

- Employees donate independently of mutual interests.
  - Employee 1 would like to donate to Greenpeace (CLUDING) or WWF (AVES).
  - Employee 2 prefers to donate to AVES or Unicef (UNICEF).
  - An efficient distribution rule would allocate both contributions to AVES.

- Employees can benefit from coordinating the donations.

**Which distribution rule should we use?**


Main Results

Which distribution rule should be chosen?

- **Goal:**
  - Guarantee (Pareto-)efficiency of the distribution.
  - Incentivize agents to donate to maximize the gains from coordination.
    → requires a strong participation axiom as contributions are initially *owned* by the agents.

- The **Nash product rule** is the only distribution rule we are aware of that simultaneously satisfies efficiency and such a strong participation axiom.
Set $N$ of agents with contributions $C = \{C_i\}_{i \in N}$ not exceeding the individual budgets $\{B_i\}_{i \in N}$.

- Christian
- Dominik
- Felix
- Florian
- Warut

Set $A$ of projects the agents can contribute to:
Set $N$ of agents with contributions $C = \{C_i\}_{i \in N}$ not exceeding the individual budgets $\{B_i\}_{i \in N}$.

I want to contribute 2

- Christian
- Dominik
- Felix
- Florian
- Warut

Set $A$ of projects the agents can contribute to:

- G
- Panda
- UNICEF
Set $N$ of agents with contributions $C = \{C_i\}_{i \in N}$ not exceeding the individual budgets $\{B_i\}_{i \in N}$.

- Christian: 2
- Dominik: 2
- Felix: 1
- Florian: 1
- Warut: 1

Set $A$ of projects the agents can contribute to:
Model

- Set $N$ of agents with contributions $C = \{ C_i \}_{i \in N}$ not exceeding the individual budgets $\{ B_i \}_{i \in N}$.

- Set $A$ of projects the agents can contribute to.

- Individual utility functions $u_i : A \rightarrow \mathbb{R}_{\geq 0}$ (here: $\rightarrow \{0, 1\}$).
- $\rightarrow$ value for one unit that is allocated to project $x$.
- Distribution $\delta : A \rightarrow \mathbb{R}_{\geq 0}$ with $\sum_{x \in A} \delta(x) = \sum_{i \in N} C_i = |C|$.
- $\rightarrow u_i(\delta) = \sum_{x \in A} \delta(x) u_i(x)$.
- Distribution rule $f$ determining the returned $\delta$. 
The Nash Product Rule

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<tr>
<th>Agents</th>
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Which distribution should be chosen?
The Nash Product Rule

Definition: Nash Product Rule

For an arbitrary profile \( C \),

\[
NASH(C) = \arg \max_{\delta \in \Delta(|C|)} \Pi_{i \in N} u_i(\delta)^{C_i}.
\]

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\( \delta_{NASH} \) | 7 | 3 | 3 | 1 | 16 |

\( \checkmark \) efficiency

\( \rightarrow \) no \( \delta' \in \Delta(|C|) \) s.t.

\( u_i(\delta') \geq u_i(\delta) \) for all \( i \in N \) and

\( u_i(\delta') > u_i(\delta) \) for some \( i \in N \).
The Nash Product Rule

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| $\delta_{NASH}$ | 7 | 3 | 3 | 1 | 16 |

- **Observation**: This distribution can be decomposed into individual distributions such that each agent only contributes to his approved projects.

- We call such distributions **decomposable**.

- Decomposability becomes very important when the distribution rule only gives recommendations to the agents.

- A NASH distribution $\delta$ can always be decomposed via $\delta_i(x) = C_i \frac{u_i(x)}{u_i(\delta)} \delta(x)$. 
The Nash Product Rule

**Definition: Nash Product Rule**

For an arbitrary profile \( C \),

\[
NASH(C) = \arg \max_{\delta \in \Delta(|C|)} \prod_{i \in N} u_i(\delta)^{C_i}.
\]

| Agents  | \( C_i \) | \( \text{G} \) | \( \text{W} \) | \( \text{E} \) | \( u_i(\delta) \) |
|---------|-----------|------------|------------|----------------|
| Christian | 2 | 2 | . | . | 3 |
| Dominik  | 2 | 1 | 1 | . | 6 |
| Felix    | 1 | . | 1 | . | 3 |
| Florian  | 1 | . | 1 | . | 3 |
| Warut    | 1 | . | . | 1 | 1 |

\( \delta_{NASH} \):

- \( \delta_{NASH} = 3 \) for efficiency
- \( \delta_{NASH} = 3 \) for decomposability

\( \delta_{NASH} = 16 \)

→ Already sufficient to ensure participation?
By not participating, i.e., saving his contribution, Felix can increase his utility gains \((u_i(\delta) - C_i)\) from coordination.

Goal: Contributing the entire budget should be a dominant strategy for each agent.

→ captured by the axiom of **contribution incentive-compatibility**.
A mechanism $f$ is **contribution incentive-compatible** if for each $i \in N$ and all profiles $C$, $u_i(f(C_{-i}, C_i)) - C_i$ is weakly increasing in $C_i$.

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Theorem

The Nash Product Rule satisfies efficiency, decomposability and contribution incentive-compatibility.

We are not aware of any other distribution rule that satisfies efficiency AND contribution incentive-compatibility!
Summary and Further Remarks

\[ NASH(C) = \arg \max_{\delta \in \Delta(|C|)} \prod_{i \in N} \left( u_i(\delta) \right)^{c_i} \]

- efficiency
- fairness
- strategy-proofness
- strong CIC or strong DEC
- impossibilities
- efficient computation
- DEC
- CIC

Matthias Greger (TUM)

Funding Public Projects

WIN 2021
NASH (C) = arg max "(|C|)
r_s = N_u (")

G

?
NASH(C) = \arg \max \prod_{\delta \in \Delta(|C|)} \left( u_i(\delta) \right)^{c_i}

Nash equilibrium is the best response to the strategy profile of others.

Efficiency
Fairness
Decentralization
CIC
Strong CIC or strong DEC
Efficient computation
Strategy-proofness
Impossibilities

Matthias Greger (TUM)
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References

