Numerical Methods for Variaional Inequalities

Proposal Thesis

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1 Background

Motivation In general, we are interested in equilibrium learning in games. There is a vast literature on learning algorithms in games (e.g. Fudenberg and Levine [1998]). But there is also stream of literature on numerical methods to solve variational stability (see Facchinei and Pang [2003]). Both problems are connected, as described for instance by Scutari et al. [2010]. The idea of the thesis is to look at certain complete information games, find the corresponding formulation as variational inequality and apply the methods from learning in games vs. methods for numerical solutions of VIs.

Model A variational inequality problem $VI(X, F)$, given a closed and convex set X and a mapping $F : X \to \mathbb{R}^n$, consists in finding $x^* \in X$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in X \tag{VI}$$

A complete information game $\Gamma = (\mathcal{I}, \mathcal{X}, u)$ consists of n bidders $\mathcal{I} = \{1, \ldots, n\}$. Each agent i can choose an action from $\mathcal{X}_i$ (with $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$) and gets a utility, depending on the action of all agents, $u_i : \mathcal{X} \to \mathbb{R}$. Assuming that the utility functions u_i are differentiable and concave in the agents own argument, i.e., $x_i \mapsto u_i(x_1, \ldots, x_i, \ldots, x_n)$ is concave, then the Nash Equilibrium x^* is characterized by

$$\langle v(x^*), x - x^* \rangle \leq 0 \quad \forall x \in \mathcal{X} \tag{NE}$$

where $v(x) = (v_i(x))_{i \in \mathcal{I}}$ with $v_i(x) = \nabla_{x_i} u_i(x)$. We can see that we can state the problem (NE) in terms of (VI). Instead of applying methods from learning in games, we want to investigate the equivalent formulation as variational inequality and investigate if there are different algorithms we could use to solve this problem.

2 Thesis

Some basic questions should be answered in the thesis

1. When can we formulate the problem of finding a Nash-equilibrium in complete information games as variational inequalities? Is this formulation equivalent?
2. What can we say about uniqueness/existence of solutions (monotonicity)?
3. Can we make use of the Merit Function (e.g. Merit Function based Algorithms)
4. What other numerical methods are there to solve variational inequalities (present 1-2 of them in detail)?
5. Maybe we can find connections between well known algorithms from the learning in games literature and numerical methods for variational inequalities.

Furthermore, we would like to look at specific example, e.g., Cournot Oligopoly ([Mertikopoulos and Zhou, 2019, Example 2.2]) to perform some numerical experiments and visualizations of the methods identified in Question 3.

The thesis would (ideally) also contain the implementation of these methods in Python.

If we have time etc. we could also think about applying these methods to the problems we are actually working on (i.e. auctions) Fichtl et al. [2022].

References

Francisco Facchinei and Jong-Shi Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, volume 1 of *Springer Series in Operations Research and Financial Engineering, Finite-Dimensional Variational Inequalities and Complementarity Problems.* Springer-Verlag, New York, 2003. ISBN 978-0-387-95580-3. doi: 10.1007/b97543.

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