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Visual Analysis of Spatial Variability and Global Correlations in Ensembles of Iso-Contours

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Supplemental Material

Integration of MVN distribution

Here, we show a proof for the identity between Eq. 4 and Eq. 5 in the paper.

Let $\mathcal{N}(\mathbf{x},\mu,\Sigma)$ denote the multivariate normal (MVN) density with mean $\mu \in \mathbb{R}^r$ and covariance matrix $\Sigma \in \mathbb{R}^{r\times r}$:

$$\mathcal{N}(\mathbf{x},\mu,\Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)\Sigma^{-1}(\mathbf{x}-\mu)\right)$$

We are interested in integrating $\mathcal{N}$ over an intersection of two slabs (= regions between two parallel hyper-planes):

$$\begin{aligned}\hat{\mathcal{S}}_a &:= \{\mathbf{x} \in \mathbb{R}^r \mid |\mathbf{a}^T\mathbf{x}+c| \le s\}\\ \hat{\mathcal{S}}_b &:= \{\mathbf{x} \in \mathbb{R}^r \mid |\mathbf{b}^T\mathbf{x}+d| \le t\}\end{aligned} \tag{1}$$

Here, we assume $\mathbf{a},\mathbf{b} \in \mathbb{R}^r$ and $c,d,s,t \in \mathbb{R}$ to be fixed parameters. Note that, for simplicity reasons, $\hat{\mathcal{S}}$ is defined slightly different to $\mathcal{S}$ in the paper (we use $\mathbf{a} = U^T\mathbf{w}_y$, $\mathbf{b} = U^T\mathbf{w}_z$, $c = \bar{\mathbf{d}}^T\mathbf{w}_z$ and $d = \bar{\mathbf{d}}^T\mathbf{w}_z$ here).

For the integration, we will perform a change of variables which aligns $\mathbf{a}$ and $\mathbf{b}$ with the first and second coordinate axis, respectively, allowing us to collapse the other dimensions of the MVN distribution. Therefore, we define the affine transformation $q(\mathbf{y}) = T\mathbf{y}+\mu$, where the $r\times r$ matrix $T := (S^{-1})^T$ is given through $S := (\mathbf{a},\mathbf{b},\mathbf{t}_3,\dots,\mathbf{t}_r)$. The columns $\mathbf{t}_i \in \mathbb{R}^r$ can be chosen arbitrarily, such that S has full rank and $\det(S) > 0$. Let $\mathbf{e}_i$ denote the i-th unit vector, then, by construction, the following holds:

$$S\mathbf{e}_1 = \mathbf{a} \wedge S\mathbf{e}_2 = \mathbf{b} \quad \Rightarrow \quad T^T\mathbf{a} = \mathbf{e}_1 \wedge T^T\mathbf{b} = \mathbf{e}_2 \tag{2}$$

To perform the change of variables, we also need to apply q^{-1} to the integration regions $\hat{\mathcal{S}}(\mathbf{a},c)$ and $\hat{\mathcal{S}}(\mathbf{b},d)$, which can be achieved by substituiting $\mathbf{x} = T\mathbf{y}+\mu$ in Eq. (1):

$$\begin{aligned}\hat{\mathcal{S}}'_a := q^{-1}(\hat{\mathcal{S}}_a) &= \{\mathbf{y} \in \mathbb{R}^r \mid |\mathbf{a}^T T\mathbf{y}+\mathbf{a}^T\mu+c| \le s\}\\ \hat{\mathcal{S}}'_b := q^{-1}(\hat{\mathcal{S}}_b) &= \{\mathbf{y} \in \mathbb{R}^r \mid |\mathbf{b}^T T\mathbf{y}+\mathbf{b}^T\mu+d| \le t\}\end{aligned}$$

Using Eq. (2), we can simplify this to

$$\begin{aligned}\hat{\mathcal{S}}'_a &= \{\mathbf{y} \in \mathbb{R}^r \mid |(\mathbf{y})_1+\mathbf{a}^T\mu+c| \le s\}\\ \hat{\mathcal{S}}'_b &= \{\mathbf{y} \in \mathbb{R}^r \mid |(\mathbf{y})_2+\mathbf{b}^T\mu+d| \le t\}.\end{aligned} \tag{3}$$

Finally, the integral which we are interested in is:

$$\begin{aligned}&\int_{\hat{\mathcal{S}}_a\cap\hat{\mathcal{S}}_b} \mathcal{N}(\mathbf{x},\mu,\Sigma)d\mathbf{x}\\ &= \frac{1}{\sqrt{(2\pi)^n\det(\Sigma)}} \int_{\hat{\mathcal{S}}_a\cap\hat{\mathcal{S}}_b} \exp\left(-\tfrac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)d\mathbf{x}\end{aligned}$$

Using the previously defined transformation $q(\mathbf{y})$, we substitute $\mathbf{x} = T\mathbf{y}+\mu$:

$$\begin{aligned}&= \frac{\det(T)}{\sqrt{(2\pi)^n\det(\Sigma)}} \int_{\hat{\mathcal{S}}'_a\cap\hat{\mathcal{S}}'_b} \exp\left(-\tfrac{1}{2}(T\mathbf{y})^T\Sigma^{-1}(T\mathbf{y})\right)\, d\mathbf{y}\\ &= \frac{1}{\sqrt{(2\pi)^n\det(T^{-1}\Sigma(T^T)^{-1})}}\\ &\qquad \cdot \int_{\hat{\mathcal{S}}'_a\cap\hat{\mathcal{S}}'_b} \exp\left(-\tfrac{1}{2}\mathbf{y}^T(T^T\Sigma^{-1}T)\mathbf{y}\right)d\mathbf{y}\\ &= \int_{\hat{\mathcal{S}}'_a\cap\hat{\mathcal{S}}'_b} \mathcal{N}\left(\mathbf{y},0,T^{-1}\Sigma(T^T)^{-1}\right)\, d\mathbf{y}\\ &= \int_{\hat{\mathcal{S}}'_a\cap\hat{\mathcal{S}}'_b} \mathcal{N}(\mathbf{y},0,S^T\Sigma S)\, d\mathbf{y}\end{aligned}$$

The integration region is now bounded by hyper-planes which are orthogonal either to the first or to the second coordinate axis (see Eq. (3)). Hence, regardless of the original dimensionality r, we can collapse all remaining dimensions of the MVN distribution, which gives the two-dimensional integral

$$= \int_{-s-\mathbf{a}^T\mu-c}^{s-\mathbf{a}^T\mu-c} \int_{-t-\mathbf{b}^T\mu-d}^{t-\mathbf{b}^T\mu-d} \mathcal{N}\left((y_1,y_2)^T,0,\Sigma'\right)\, dy_2\, dy_1,$$

with integrations bounds according to Eq. (3) and with $\Sigma' \in \mathbb{R}^{2\times 2}$ being the upper left 2×2 submatrix of $S^T\Sigma S$:

$$\Sigma' = \begin{pmatrix} \mathbf{a}^T\Sigma\mathbf{a} & \mathbf{a}^T\Sigma\mathbf{b}\\ \mathbf{a}^T\Sigma\mathbf{b} & \mathbf{b}^T\Sigma\mathbf{b}\end{pmatrix}.$$

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This is the expression we were looking for. Additionally, we can transform it into a normalized ρ-form by rescaling along both coordinate axes. This is the format that is typically accepted by libraries which can calculate 2D rectangular normal probabilities:

$$\int_{\hat{\mathcal{S}}_a \cap \hat{\mathcal{S}}_b} \mathcal{N}(\mathbf{x}, \mu, \Sigma) d\mathbf{x}$$

$$= \int_{\frac{-s-\mathbf{a}^T\mu-c}{\mathbf{a}^T\Sigma\mathbf{a}}}^{\frac{s-\mathbf{a}^T\mu-c}{\mathbf{a}^T\Sigma\mathbf{a}}} \int_{\frac{-t-\mathbf{b}^T\mu-d}{\mathbf{b}^T\Sigma\mathbf{b}}}^{\frac{t-\mathbf{b}^T\mu-d}{\mathbf{b}^T\Sigma\mathbf{b}}} \mathcal{N}\left((y_1, y_2)^T, 0, \Sigma''\right) \, dy_2 \, dy_1,$$

$$\Sigma'' = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \quad \text{with} \quad \rho = \frac{\mathbf{a}^T\Sigma\mathbf{b}}{\sqrt{(\mathbf{a}^T\Sigma\mathbf{a}) \cdot (\mathbf{b}^T\Sigma\mathbf{b})}}.$$

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