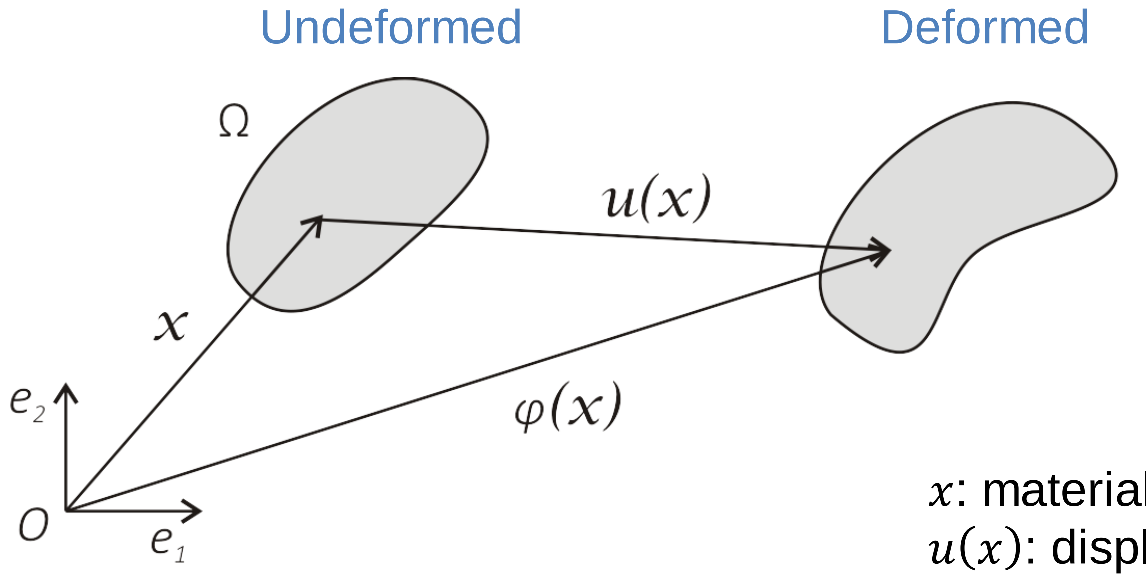
Outline

follows the structure of the report

- Introduction
- Mesh-based Modeling of Cuts
- Finite Element Simulation for Virtual Cutting
  - Extended FEM
  - Composite FEM
  - Polyhedral FEM
  - Discussion on FEMs
- Numerical Solvers
- Meshfree Methods
- Summary & Application Study
- Discussion & Conclusion

Physically-based Deformation Models

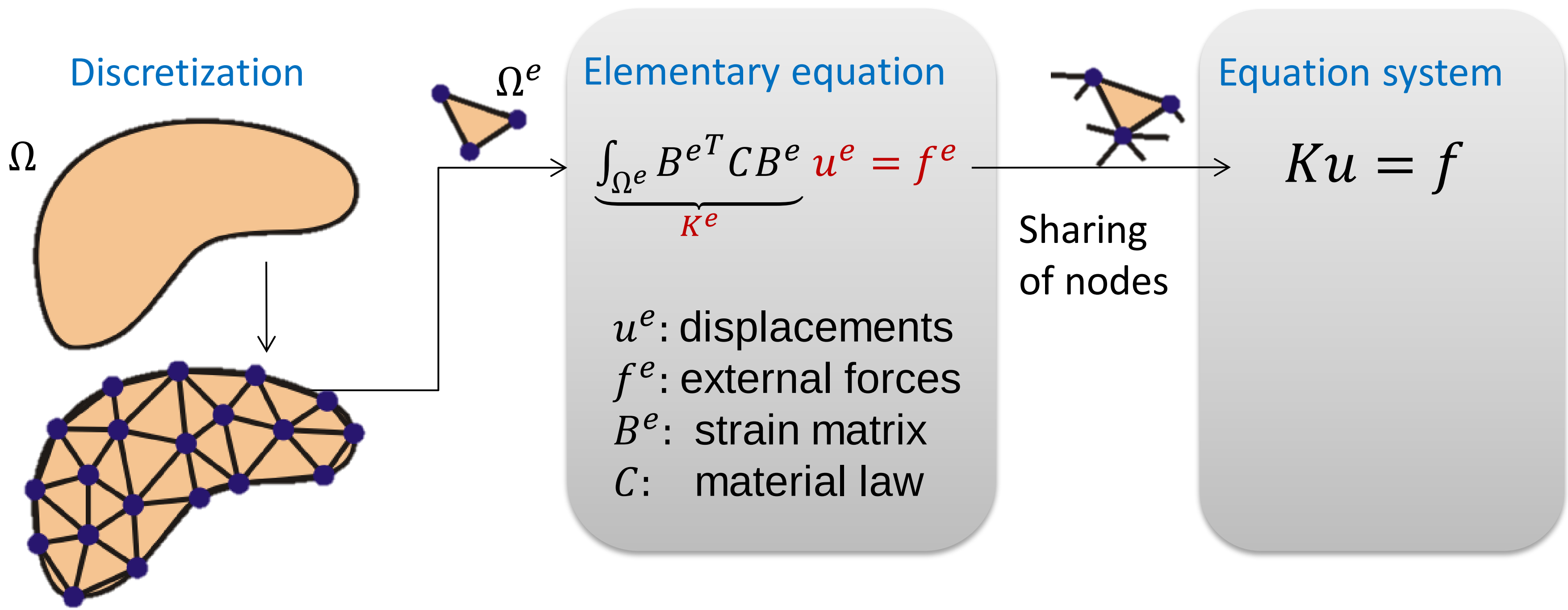
- Compute the object's deformation due to external forces
  - Introduced to computer graphics by Terzopoulos et al. 1987
  - Surveyed in STAR by Nealen et al. 2006
- Finite element methods (FEM), meshfree methods, mass-spring systems, etc.



x: material coordinates
$u(x)$: displacement field
$\varphi(x)$: deformation field

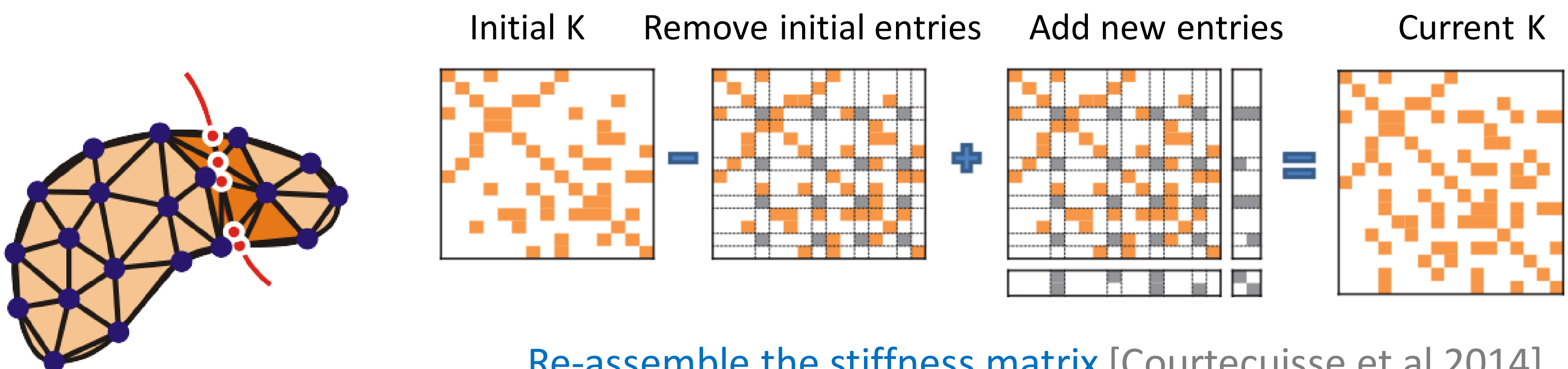
Recap: Finite Element Simulation of Elasticity

1) Discretize the object into elements
2) Build elementary equations $K^e u^e = f^e$
3) Assemble a linear system of equations $Ku = f$
4) Solve for the displacement field u



Virtual Cutting Using the Standard FEM

1) Split elements which are touched by the scalpel
2) Re-build elementary equations $K^e u^e = f^e$
3) Re-assemble a linear system of equations $Ku = f$
   - Remove entries of the deleted initial elements
   - Add entries of the split new elements
4) Solve for the displacement field u

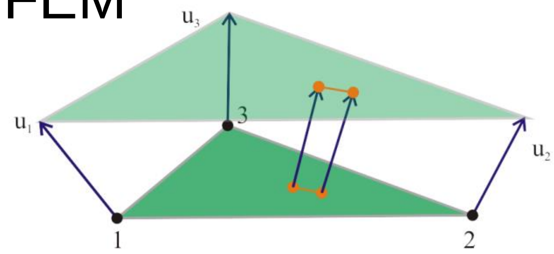


Re-assemble the stiffness matrix [Courtecuisse et al 2014]

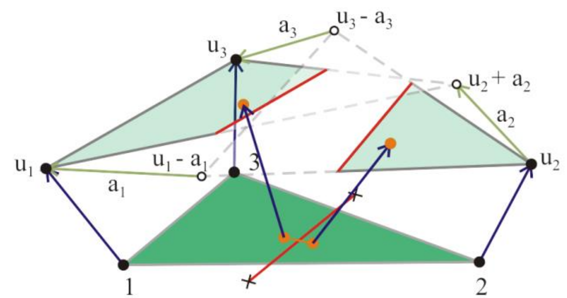
The Extended Finite Element Method (XFEM)

- Model material discontinuities by enriching the basis functions of the solution space [Belytschko et al. 1999]
  - Adapting basis functions instead of modifying the meshes

- Displacement field $u(x)$ in the standard FEM
- $u(x) = \Phi^e(x)\, u^e$
  - $\Phi^e(x)$: shape matrix
  - u^e: displacement vector at nodes

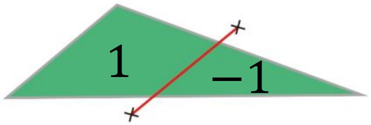


- Displacement field $u(x)$ in the extended FEM
- $u(x) = \Phi^e(x)\, u^e + \Psi^e(x)\Phi^e(x)\, a^e$
  - $\Psi^e(x)$: shape enrichment matrix
  - a^e: added displacement vector at nodes

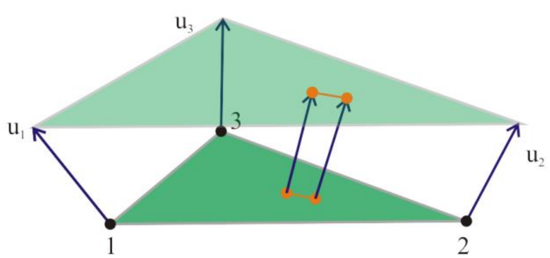


XFEM – Discontinuous Enrichment Function

- $u(x) = \Phi^e(x)\, u^e + \Psi^e(x)\Phi^e(x)\, a^e$
- Shifted enrichment function
  - $\psi_i^e(x) = \frac{H(x) - H(x_i)}{2}$
  - $H(x) = \begin{cases} 1, & \text{if } x \text{ is on the cut's left side;} \\ -1, & \text{if } x \text{ is on the cut's right side.} \end{cases}$



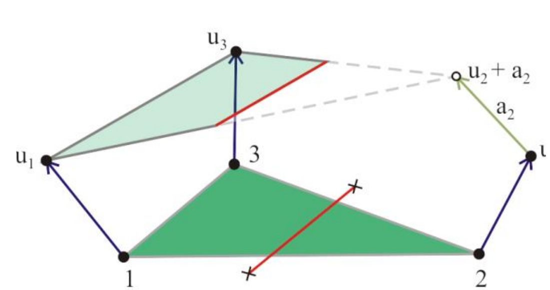
Heaviside function $H(x)$



Both left and right sides:

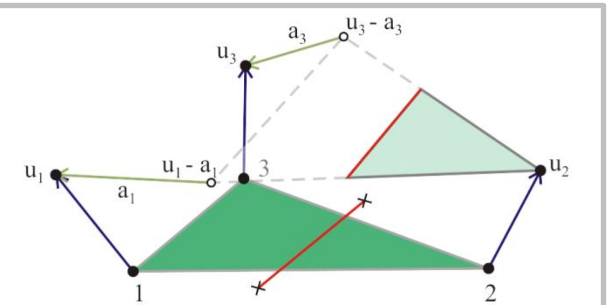
$$u(x) = \Phi^e(x)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Standard FEM



Left side of the triangle:

$$u(x) = \Phi^e(x)\begin{pmatrix} u_1 \\ u_2 + a_2 \\ u_3 \end{pmatrix}$$



Right side of the triangle:

$$u(x) = \Phi^e(x)\begin{pmatrix} u_1 - a_1 \\ u_2 \\ u_3 - a_3 \end{pmatrix}$$

Extended FEM

XFEM - Stiffness Matrices

- Standard stiffness matrix $K^e := \int_{\Omega^e} {B^e}^T C B^e$
  - Material law C relates strain to stress $\sigma = C : \epsilon$
  - Strain matrix $B^e = \left(B_1^e, \dots, B_{n_v}^e\right)$

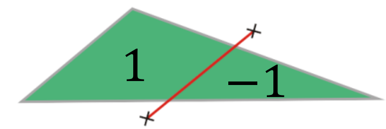
- Enriched stiffness matrix ${}^xK^e = \int_{\Omega^e} ({}^xB^e)^T \, C \, {}^xB^e dx$
- ${}^xB^e = \left(B_1^e, \dots, B_{n_v}^e, \ \psi_1^e B_1^e, \dots, \psi_{n_v}^e B_{n_v}^e\right)$
- ${}^xK^e = \begin{pmatrix} K^{e,uu} & K^{e,ua} \\ K^{e,au} & K^{e,aa} \end{pmatrix}$

XFEM – Detailed Cutting of Shells

- Store enrichment function as a 2D texture

Enrichment texture within a quad mesh

Simulation result



Heaviside function $H(x)$

$$H(x) = \begin{cases} 1, & \text{on left} \\ -1, & \text{on right} \end{cases}$$

[Kaufmann et al 2009]

XFEM – Detailed Cutting of Shells

- Store enrichment function as a 2D texture

Enrichment texture
within a quad mesh

Simulation result

Multiple cuts

[Kaufmann et al 2009]

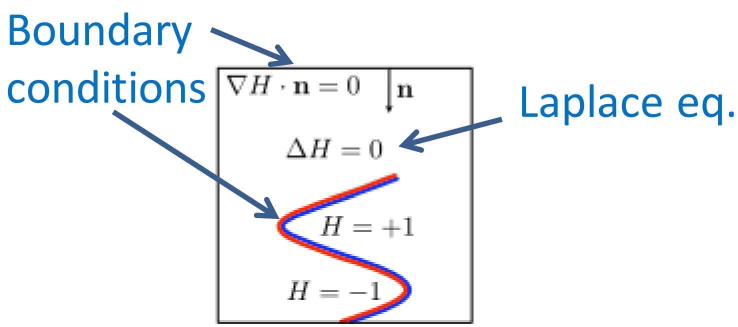
XFEM – Detailed Cutting of Shells

- Store enrichment function as a 2D texture
- Employ harmonic enrichment function for partial cuts

Enrichment texture within a quad mesh

Simulation result

Enrichment texture of a partial cut



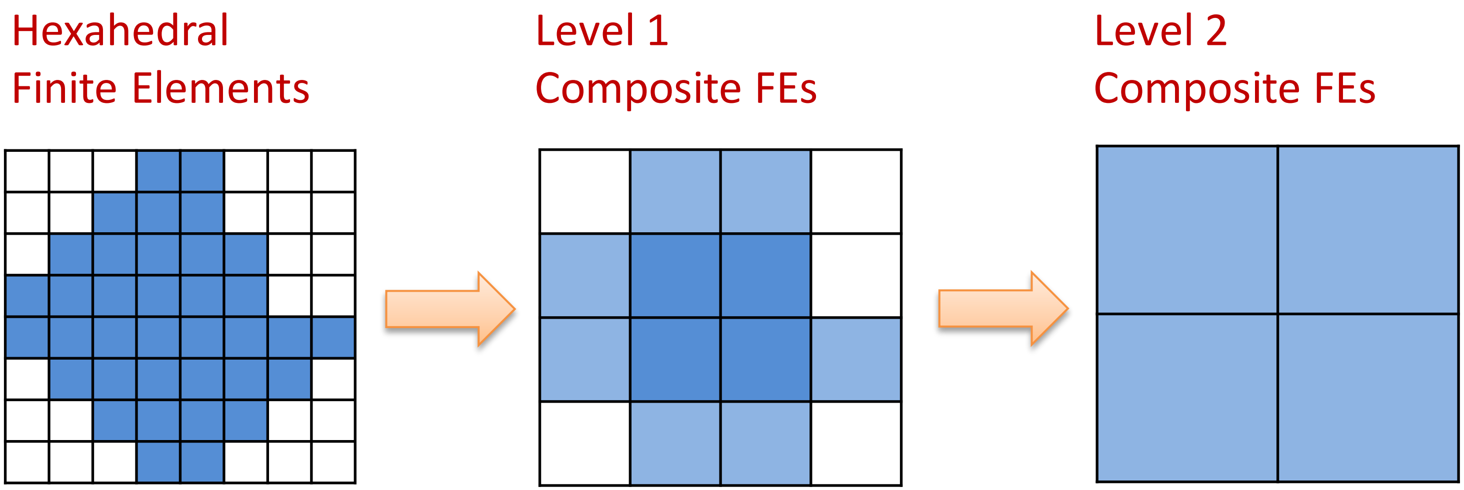
Harmonic enrichments $H(x)$

Simulation result

[Kaufmann et al 2009]

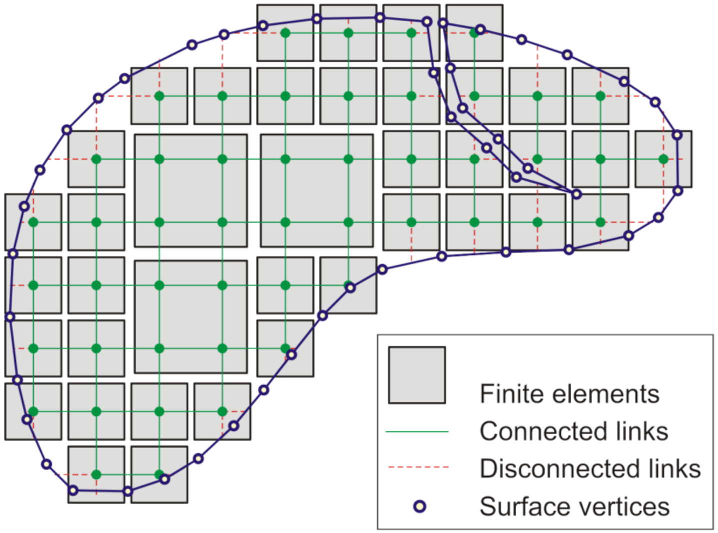
The Composite Finite Element Method (CFEM)

- Approximate a high resolution finite element discretization by a small set of coarser elements [Hackbusch & Sauter 1997]
  - Reduce the number of simulation DOFs
  - Also used for: construct a grid hierarchy for the multigrid solver

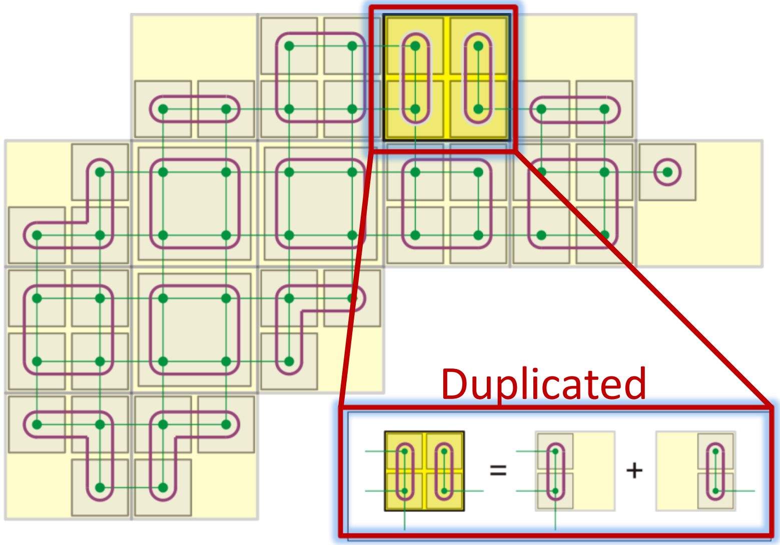


CFEM – Geometrical & Topological Composition

- Duplicated elements: Each connected part is merged to one independent element
  - Located at the same place in the reference configuration
  - But have different topology connections



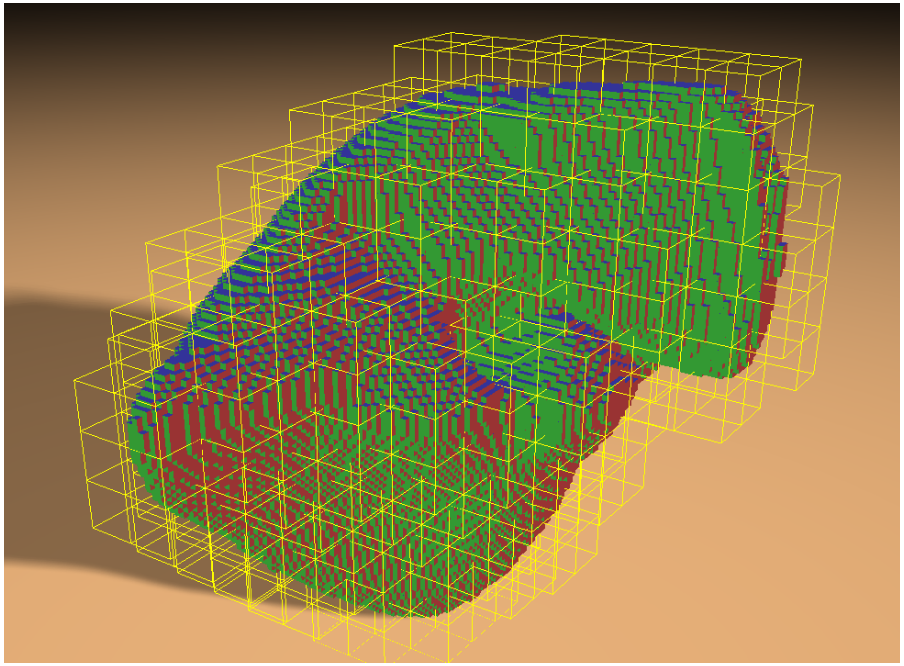
Linked octree representation



Composite finite element

CFEM – Geometrical & Topological Composition

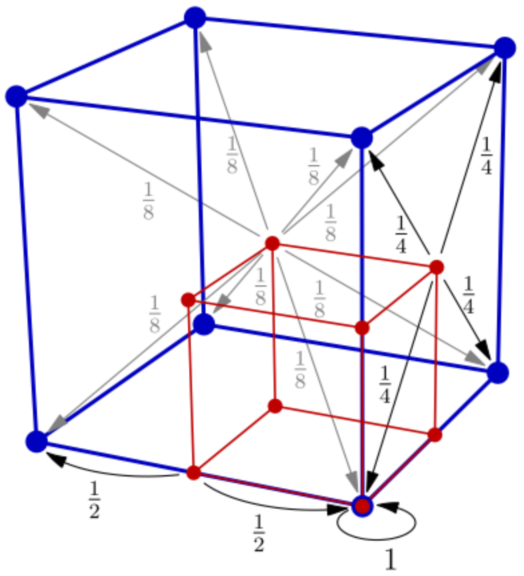
- Duplicated elements: Each connected part is merged to one independent element
  - Located at the same place in the reference configuration
  - But have different topology connections
- Iteratively merge blocks of 2^3 elements into 1 element



Fine resolution: 82×83×100
Composition level: 3 (8^3->1)

CFEM – Numerical Composition

- Displacement interpolation
  - composite elements → fine hexahedra
  - $u = I\,\tilde{u}$
- Stiffness matrix assembly
  - fine hexahedra → composite elements
  - $\widetilde{K} = I^{\mathrm{T}} K I$

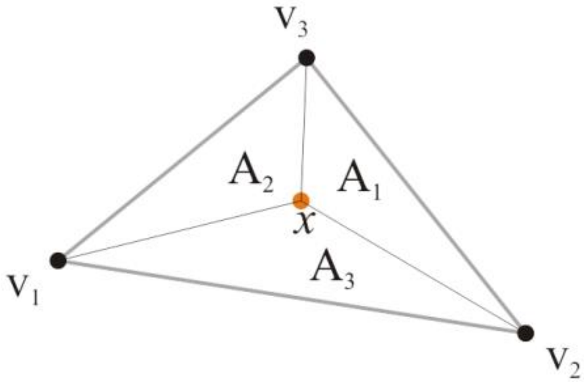


- $\widetilde{K}^{c}_{mn} = \sum_{e \text{ in } c} \sum_{i=1}^{8} \sum_{j=1}^{8} w^{c\to e}_{m\to i} w^{c\to e}_{n\to j} K^{e}_{ij}\,, \quad m, n = 1, \dots, 8$
- $w^{c\to e}_{m\to i} = \left(1 - \frac{\left|x^{c}_{m} - x^{e}_{j}\right|}{s^{c}}\right)\left(1 - \frac{\left|y^{c}_{m} - y^{e}_{j}\right|}{s^{c}}\right)\left(1 - \frac{\left|z^{c}_{m} - z^{e}_{j}\right|}{s^{c}}\right)$

The Polyhedral Finite Element Method (PFEM)

- Directly evaluate deformation on general polyhedra [Wicke et al. 2007]
  – Tetrahedralization/hexahedralization process is avoided

- Shape functions: $u(x) = \sum_{i=1}^{n_v} \phi_i(x)\, u_i$
  – Tetrahedron: barycentric interpolation
  – Hexahedron: tri-linear interpolation
  – Polyhedron: ??



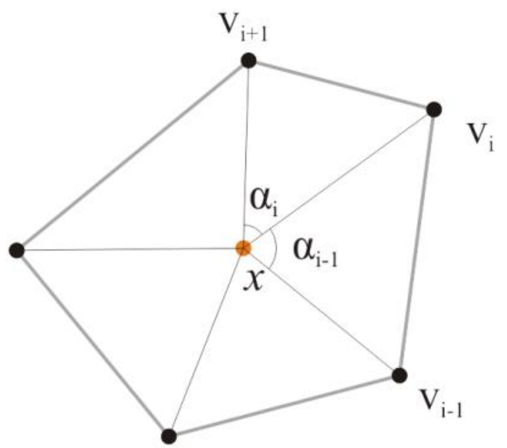
$$\phi_i(x) = \frac{A_i}{\sum_{j=1}^{n_v} A_j} u_i,$$

$$A_i = A(x, v_{i-1}, v_{i+1})$$

Barycentric interpolation for a triangle

PFEM – Shape Functions

- Mean value interpolation function
  - Generalization of barycentric interpolation to convex polyhedra
- Shape functions: $u(x) = \sum_{i=1}^{n_v} \phi_i(x)\, u_i$
  - Kronecker delta property: $\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
  - Completeness: $\sum_{i=1}^{n_v} \phi_i(x) = 1$



$$\phi_i(x) = \frac{w_i}{\sum_{j=1}^{n_v} w_j}$$

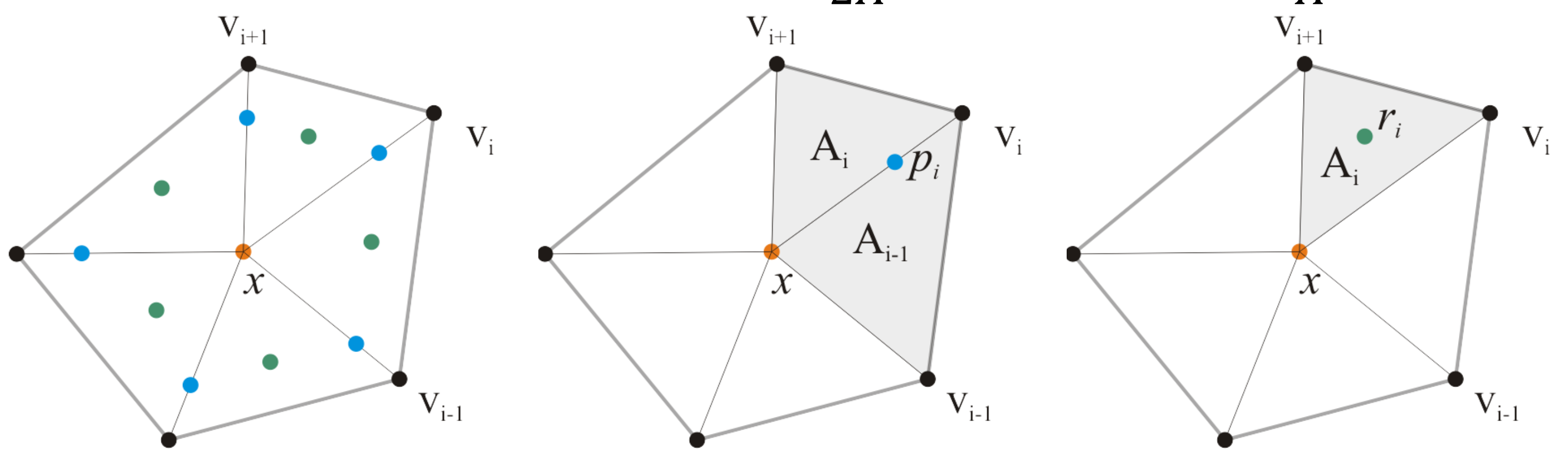
$$w_i = \frac{\tan\left(\alpha_{i-1}/2\right) + \tan\left(\alpha_i/2\right)}{||v_i - x||}$$

Mean value interpolation for a polygon

PFEM – Stiffness Matrices

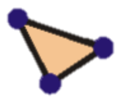
- Stiffness matrix $K^e \coloneqq \int_{\Omega^e} {B^e}^T C B^e$
  - Analytical integration over general polyhedra is non-trivial
  - Approximated by numerical integration at a few samples
- $K^e = \sum_i \frac{\mu_i^e}{2} (B^e(p_i))^{\mathrm{T}}\, C\, B^e(p_i) + \sum_i \frac{\kappa_i^e}{2} (B^e(r_i))^{\mathrm{T}}\, C\, B^e(r_i)$

$$\mu_i^e = \frac{A_{i-1} + A_i}{2A^e} \qquad \kappa_i^e = \frac{A_i}{A^e}$$



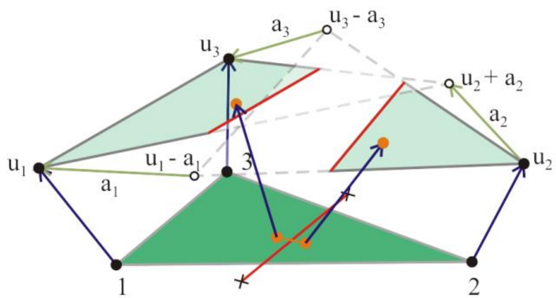
Discussion on FEMs

- Standard FEM
  - Each spatial mesh maps to one specific computational finite element



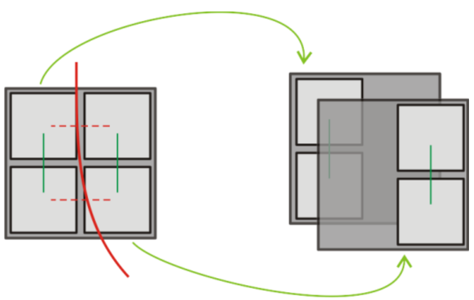
$$K^e u^e = f^e$$

- Extended FEM, composite FEM
  - Disconnected spatial mesh corresponds to multiple, duplicated simulation DOFs



$$\begin{pmatrix} K^{e,uu} & K^{e,ua} \\ K^{e,au} & K^{e,aa} \end{pmatrix} \begin{pmatrix} u^e \\ a^e \end{pmatrix} = \begin{pmatrix} f^e \\ 0 \end{pmatrix}$$

Extended FEM



Composite FEM